Note: This implies, if X complete, residue set is dense
(E empty interior $\Rightarrow$ $X \setminus E$ dense)

Recall that E is closed nowhere dense set
$\Leftrightarrow$ $X \setminus E$ $(= X \setminus \overline{E})$ is an open dense set.

Hence Thm 4.9 can be rephased as

Thm 4.9′ (Baire Category Theorem)
In a complete metric space, countable intersection of open dense sets is dense.

i.e. If (X, d) is complete and $G_n \subset X$ is a sequence of open dense sets in X, then $\bigcap_{n=1}^{\infty} G_n$ is dense.

(Pf: Ex!)

Cor 4.10: Let (X, d) be complete. Suppose that $X = \bigcup_{n=1}^{\infty} E_n$ with E_n are closed subsets. Then at least one of these E_n's has non-empty interior.

Pf: Suppose not, then all E_n has empty interior.
$\Rightarrow$ E_n is nowhere dense, $\forall n$.
Hence $X = \bigcup_{n=1}^{\infty} E_n$ is of 1^{st} category.
Baire Category Thm $\Rightarrow$ X has empty interior which is a contradiction since $X^{\circ} = X$. ※

Remark: This corollary implies that it is impossible to decompose a complete metric space into a countable union of nowhere dense sets.

(i.e. complete metric space itself is of 2^{nd} category.)

$\boxed{\text{Cor 4.11 A set of } 1^{st} \text{ category in a complete metric space cannot be a residual set, and vice versa.}}$

($\Rightarrow$ residual sets of a complete metric space is of 2^{nd} category)

Pf: Let E be a set of 1^{st} category,

then $E = \bigcup_{n=1}^{\infty} E_n$ with E_n nowhere dense.

If E is also a residual set, then $X \setminus E$ is also of 1^{st} category, hence

$$X \setminus E = \bigcup_{n=1}^{\infty} E_n' \quad \text{with } E_n' \text{ nowhere dense.}$$

$$\Rightarrow X = E \cup (X \setminus E) = \left(\bigcup_{n=1}^{\infty} E_n\right) \cup \left(\bigcup_{n=1}^{\infty} E_n'\right)$$

Taking closure of E_n & E_n', $X \subset \left(\bigcup_{n=1}^{\infty} \overline{E_n}\right) \cup \left(\bigcup_{n=1}^{\infty} \overline{E_n'}\right) \ (\subset X)$

$$\Rightarrow \quad X = \left(\bigcup_{n=1}^{\infty} \overline{E_n}\right) \cup \left(\bigcup_{n=1}^{\infty} \overline{E_n'}\right)$$

i.e. X is a countable union of close subsets with empty interiors. This contradicts Cor 4.10.

The other way is similar. ※

eg: $\mathbb{R}$ is complete, $\mathbb{Q}$ of 1^{st} category $\Rightarrow \mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ is of 2^{nd} category.

Applications of Baire Category Theorem (to function spaces)

Thm 4.13 The set of all continuous, nowhere differentiable functions forms a residual set in $C[a,b]$ and hence dense in $C[a,b]$.

To prove the theorem, we need a lemma:

Lemma 4.2: Let $f \in C[a,b]$ be differentiable at x. Then it is Lipschitz continuous at x

(It is clear near x. The main issue is for points not near x)

Pf: By assumption ($\forall \varepsilon > 0$, say $\varepsilon = 1$) $\exists \delta_0 > 0$ such that $\forall y \in (x-\delta_0, x+\delta_0) \setminus \{x\}$ (& $y \in [a,b]$)

$$\left| \frac{f(y)-f(x)}{y-x} - f'(x) \right| < 1 .$$

$$\Rightarrow \quad |f(y)-f(x)| \leq (1+|f'(x)|)|y-x|$$

$$\forall y \in (x-\delta_0, x+\delta_0) \cap [a,b]$$

If $[a,b] \setminus (x-\delta_0, x+\delta_0) = \phi$, we are done.

If not, then for $y \in [a,b] \setminus (x-\delta_0, x+\delta_0)$,

$$|y-x| \geq \delta_0$$

and hence

$$|f(y)-f(x)| \leq |f(y)| + |f(x)| \leq 2\|f\|_\infty \leq \frac{2\|f\|_\infty}{\delta_0} |y-x|$$

Let $L = \max\{1 + |f'(x)|, \frac{2\|f\|_\infty}{\delta_0}\}$, we have

$$|f(y) - f(x)| \leq L|y - x|, \quad \forall y \in [a,b]. \quad \#$$

<u>Pf of Thm 4.13</u>

We only need to show the case $[a,b] = [0,1]$.

$\forall L > 0$, define

$$S_L = \left\{ f \in C[0,1] : \begin{array}{l} f \text{ is Lip. cts at some } x \in [0,1] \\ \text{with Lip. Const.} \leq L \end{array} \right\}$$

<u>Claim 1</u> : S_L is closed.

<u>Pf</u>: Let $\{f_n\}$ be a seq. in S_L which converges to some $f \in C[0,1]$ in d_∞ metric.

By definition of S_L, $\forall n \geq 1$

$\exists x_n \in [0,1]$ such that

f_n is Lip. cts at x_n with Lip const $\leq L$

i.e. $|f_n(y) - f_n(x_n)| \leq L|y - x_n|, \quad \forall y \in [0,1]$.

We may assume that $x_n \to x^*$ for some $x^* \in [0,1]$ by passing to a subseq. (The corresponding subseq. f_n is still convergent & $f_n \to f$ in d_∞)

Then $|f(y) - f(x^*)| \leq |f(y) - f_n(y)| + |f_n(y) - f(x^*)|$

$$\le \|f-f_n\|_\infty + |f_n(y)-f_n(x_n)| + |f_n(x_n)-f(x^*)|$$
$$\le \|f-f_n\|_\infty + L|y-x_n| + |f_n(x_n)-f_n(x^*)| + |f_n(x^*)-f(x^*)|$$
$$\le 2\|f-f_n\|_\infty + L|y-x_n| + L|x_n-x^*|$$
$$\le 2\|f-f_n\|_\infty + L|y-x^*| + L|x^*-x_n| + L|x_n-x^*|$$
$$= L|y-x^*| + 2(\|f-f_n\|_\infty + L|x_n-x^*|)$$

Letting $n\to+\infty$, we have

$$|f(y)-f(x^*)| \le L|y-x^*| \quad , \quad \forall y\in[0,1]$$

$\Rightarrow f\in S_L$. ※

<u>Claim</u> 2 : S_L is nowhere dense ($C[0,1]\setminus S_L$ is dense)

<u>Pf</u>: Let $f\in S_L$

By Weierstrass Approximation Theorem,

$\forall \varepsilon>0$, $\exists$ a polynomial p such that

$$\|f-p\|_\infty < \varepsilon/2$$

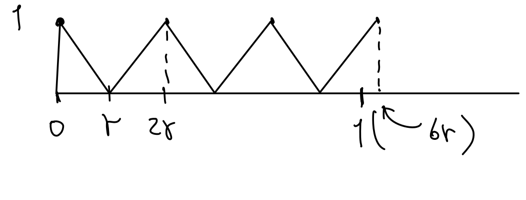
Let the Lip. constant of p be L_1.

For $r>0$ ($r<1$, not necessary rational),

let

$\varphi(x)$ be the restriction to $[0,1]$ of the jig-saw function of period $2r$ satisfying $\varphi(0)=1$, $0\le\varphi\le1$, and slope of the graph of φ is $\pm\frac{1}{r}$

(except the finitely many non-differentiabe points.)



Then consider the function

$$g(x) = p(x) + \frac{\varepsilon}{2}\varphi(x) \in C[0,1]$$

Then $\|g-f\|_\infty \leq \|p-f\|_\infty + \frac{\varepsilon}{2}\|\varphi\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

On the other hand

$$\left|\frac{\varepsilon}{2}\varphi(y) - \frac{\varepsilon}{2}\varphi(x)\right| \leq |g(y)-g(x)| + |p(y)-p(x)|$$

$$\Rightarrow \quad \frac{\varepsilon}{2}|\varphi(y)-\varphi(x)| \leq |g(y)-g(x)| + L_1|y-x|.$$

Note that $\forall x \in [0,1]$, $\exists y \in [0,1]$ near x such that

$$|\varphi(y)-\varphi(x)| = \frac{1}{r}|y-x|$$

$$\Rightarrow \quad |g(y)-g(x)| \geq \left(\frac{\varepsilon}{2r} - L_1\right)|y-x|$$

Hence if we choose $r < \dfrac{\varepsilon}{2(L+L_1)}$, then

$\forall x \in [0,1]$, $\exists y \in [0,1]$ such that

$$|g(y)-g(x)| \geq \left(\frac{\varepsilon}{2r} - L_1\right)|y-x| > L|y-x|.$$

i.e. $\forall x \in [0,1]$, g is not Lip. cts at x with Lip constant L.

$$\Rightarrow \quad g \notin S_L$$

We have proved that $\forall f \in S_L$, $\forall \varepsilon > 0$, $B_\varepsilon^\infty(f) \setminus S_L \neq \emptyset$.
By claim 1, S_L is closed hence S_L is nowhere dense.

#

Final Step:

Let $S = \{ f \in C[0,1] : f \text{ is differentiable at some } x \in [0,1] \}$

Then by Lemma 4.12, $\forall f \in S$, $f \in S_N$ for some $N \in \mathbb{N}$.

$\Rightarrow \quad S \subset \bigcup_{N=1}^{\infty} S_N$.

By claim 2, S is of 1^{st} category.

And Baire Category Thm (using $C[0,1]$ is complete)

$\Rightarrow$ S has empty interior.

$\Rightarrow$ Set of cts, but nowhere differentiable functions on $[0,1]$ ($=$ complement of S in $C[0,1]$) is a residual set and dense in $C[0,1]$. ✗

Remarks (i) The Thm and its proof provide no explicit example, not even a method to construct a continuous nowhere differentiable function.

(ii) An explicit example was given by Weierstrass:

$$W(x) = \sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n} \quad \text{on } \mathbb{R}$$

(it comes from Fourier series, actually Weierstrass provided a family.)

Further examples

Def Let $f:[a,b]\to\mathbb{R}$ be a function, and

$$L:\mathbb{R}\to\mathbb{R},\quad x\mapsto \alpha x+\beta \quad \text{for some } \alpha,\beta\in\mathbb{R}$$

(L is degree ≤ 1 poly)

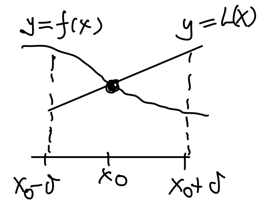
We say <u>L crosses f</u> (or <u>f crosses L</u>)

if $\exists\, x_0\in[a,b]$, and $\delta>0$ such that

$$f(x_0)=L(x_0)$$

and either one of following holds

(i) $\begin{cases} L(x)\le f(x), & \forall x\in[x_0-\delta,x_0]\cap[a,b] \\ L(x)\ge f(x), & \forall x\in[x_0,x_0+\delta]\cap[a,b]\end{cases}$



(ii) $\begin{cases} L(x)\ge f(x), & \forall x\in[x_0-\delta,x_0]\cap[a,b] \\ L(x)\le f(x), & \forall x\in[x_0,x_0+\delta]\cap[a,b]\end{cases}$