<u>Note</u>: E is nowhere dense $\Leftrightarrow$ $X \setminus \overline{E}$ is dense in X

<u>Pf</u>: E is nowhere dense

$\Leftrightarrow$ $\forall x \in X$ & $r>0$, $B_r(x) \not\subseteq \overline{E}$ (since $\overline{E}$ contains no ball)

$\Leftrightarrow$ $\forall x \in X$ & $r>0$, $B_r(x) \cap (X \setminus \overline{E}) \neq \emptyset$

$\Leftrightarrow$ $X \setminus \overline{E}$ is dense.

<u>Def</u>: Let (X, d) be a metric space. A point $x \in X$ is called an <u>isolated point</u> if $\{x\}$ is <u>open</u> in X.

<u>Notes</u>:
- As $\{x\}$ is always closed in a metric space, $\{x\}$ is both <u>open and closed</u> in X $\Leftrightarrow$ x is an isolated point.
- x isolated $\Rightarrow$ $\{x\}$ is <u>not</u> nowhere dense.

<u>egs</u>:
- $\mathbb{R}$ has no isolated points (since $\{x\}$ is not open in $\mathbb{R}$, $\forall x \in \mathbb{R}$)
- All points in $\mathbb{Z}$ (subspace of $\mathbb{R}$) are isolated in $\mathbb{Z}$ (not $\mathbb{R}$).

  since: $\forall n \in \mathbb{Z}$, $\{n\} = B_{\frac{1}{2}}(n)$ metric ball in $\mathbb{Z}$

  (But $(\mathbb{Z}$, subspace metric$) \neq (\mathbb{Z}$, discrete metric$)$

  unbounded   bounded ≤ 1

Prop 4.7 Let (X, d) be a metric space.

(a) E is nowhere dense in X $\Rightarrow$

(i) $\bar{E}$ is nowhere dense in X;

(ii) if $E' \subset E \Rightarrow E'$ is nowhere dense in X

(b) The union of finite many nowhere dense sets (in X) is nowhere dense (in X)

(c) If (X, d) has <u>no</u> isolated point, then every finite set is nowhere dense.

Pf: (a) Trivial (HW7)

(b) Let E_1, E_2 be nowhere dense sets

Then $G_1 = X \setminus \bar{E_1}$ and $G_2 = X \setminus \bar{E_2}$ are <u>open dense</u> set.

Clearly $G_1 \cap G_2$ is open.

<u>claim</u>: $G_1 \cap G_2$ is dense in X.

Pf: $\forall x \in X$ & $r > 0$,

G_1 dense $\Rightarrow B_r(x) \cap G_1 \neq \phi$

$\Rightarrow \exists\, x_1 \in B_r(x) \cap G_1$.

Since $B_r(x) \cap G_1$ is open, $\exists\, \rho > 0$ such that

$B_\rho(x_1) \subset B_r(x) \cap G_1$.

Now G_2 dense $\Rightarrow B_\rho(x_1) \cap G_2 \neq \phi$

$\Rightarrow B_r(x) \cap (G_1 \cap G_2) \supset B_\rho(x_1) \cap G_2 \neq \phi$

This proves the claim.

Hence $X \setminus (G_1 \cap G_2) = (X \setminus G_1) \cup (X \setminus G_2) = \bar{E_1} \cup \bar{E_2}$ is nowhere dense.

By (a)(ii), $E_1 \cup E_2 \subset \overline{E_1} \cup \overline{E_2}$

$\Rightarrow$ $E_1 \cup E_2$ is also nowhere dense.

Then, induction $\Rightarrow$ $\bigcup_{i=1}^{k} E_i$ is nowhere dense provided $E_1, \cdots, E_k$ are nowhere dense.

(c) Assume (X, d) has no isolated point,

<u>Claim</u>: $\forall x \in X$, $\{x\}$ is nowhere dense in X.

<u>Pf</u>: Suppose not, then $\overline{\{x\}}$ $(=\{x\})$ contains some open ball $B_r(y)$, ie $B_r(y) \subset \overline{\{x\}} = \{x\}$

This implies, $y = x$ &

$$B_r(x) \subset \{x\} \subset B_r(x).$$

$\Rightarrow$ $\{x\} = B_r(x)$ is open

$\Rightarrow$ x is isolated which is a contradiction.

Then by (b) & the claim, any finite set is nowhere dense. ※

<u>eg</u>: $(\mathbb{R}, d(x,y) = |x-y|)$ has no isolated point

$\Rightarrow$ any $\{x_1, \cdots, x_n\}$ is nowhere dense.

But for countable subsets, we have no such conclusion:

- $\mathbb{N} = \{1, 2, 3, \cdots\}$ countable and nowhere dense (Ex!)
- $\mathbb{Q}$ countable, but <u>not</u> nowhere dense. (in fact, $\mathbb{Q}$ is dense)

Examples in infinite dimensional normed spaces

eg: Let $M[a,b]$ = space of bounded functions on $[a,b]$.

Then $\|f\|_\infty = \sup_{[a,b]} |f(x)|$ is well-defined and is a norm on $M[a,b]$.

Clearly $(C[a,b], d_\infty)$ is a metric (also vector) subspace of $(M[a,b], d_\infty)$

<u>Claim</u>: $C[a,b]$ is nowhere dense in $M[a,b]$ (wrt d_∞ metric).

<u>Pf</u>: (1) Clearly, $C[a,b]$ is closed in $M[a,b]$

(uniform limit of cts. functions is cts.)

Hence $C[a,b]$ is nowhere dense in $M[a,b]$

$\Leftrightarrow$ $M[a,b] \setminus \overline{C[a,b]} = M[a,b] \setminus C[a,b]$ is dense.

$\therefore$ We only need to show that

(2) $\forall$ $B^\infty_\varepsilon(f) \subset M[a,b]$, $B^\infty_\varepsilon(f) \cap \big(M[a,b] \setminus C[a,b]\big) \neq \emptyset$.

(i) If $f \in M[a,b] \setminus C[a,b]$, we are done.

(ii) If $f \in C[a,b]$,

define $$g(x) = \begin{cases} f(x) + \frac{\varepsilon}{2}, & x \in [a,b] \cap \mathbb{Q} \\ f(x) - \frac{\varepsilon}{2}, & x \in [a,b] \setminus \mathbb{Q}. \end{cases}$$

Then $g(x) - f(x) = \pm \frac{\varepsilon}{2}$

$\Rightarrow$ $\|g - f\|_\infty = \frac{\varepsilon}{2}$ $\Rightarrow$ $g \in B^\infty_\varepsilon(f)$

If $g \in C[a,b]$, then $g - f = \begin{cases} \varepsilon/2, & [a,b] \cap \mathbb{Q} \\ -\varepsilon/2, & [a,b] \setminus \mathbb{Q} \end{cases}$

is continuous, which is impossible. Hence $g \in M[a,b] \setminus C[a,b]$

$\Rightarrow$ $B^\infty_\varepsilon(f) \cap (M[a,b] \setminus C[a,b]) \neq \emptyset$ ※

eg: Let l_∞ = space of bounded sequences with d_∞ metric

$$d_\infty(x,y) = \sup_n |x_n - y_n| \quad \text{for } x = \{x_n\},\ y = \{y_n\}$$

Let $\mathcal{C}$ = subset of convergent sequences.

Then $\mathcal{C}$ is nowhere dense in (l_∞, d_∞).

Pf: We only need to show (1) & (2) in the following

(1) $\mathcal{C}$ is closed in l_∞.

Pf: (We'll show that $l_\infty \setminus \mathcal{C}$ is open)

Let $x = \{x_n\} \in l_\infty \setminus \mathcal{C}$

Then x_n diverges and

$$(+\infty >)\ L = \limsup_n x_n > \liminf_n x_n = l\ (> -\infty)$$

Take $\varepsilon = \frac{L-l}{3} > 0$

then $\forall\, y = \{y_n\} \in B^\infty_\varepsilon(x)$, we have

$$x_n - \varepsilon < y_n < x_n + \varepsilon, \quad \forall n$$

$$\begin{cases} \limsup x_n - \varepsilon \le \limsup y_n \\ \liminf y_n \le \liminf x_n + \varepsilon \end{cases}$$

$$\Rightarrow \quad \limsup y_n \ge L - \varepsilon = \frac{2L+l}{3} > \frac{L+2l}{3} \quad (\text{since } L > l)$$

$$= l + \varepsilon \ge \liminf y_n$$

$\Rightarrow$ $y = \{y_n\}$ is divergent.

Hence $B^\infty_\varepsilon(x) \subset l_\infty \setminus \mathcal{C}$ $\Rightarrow$ $l_\infty \setminus \mathcal{C}$ is open

& this proves (1)

(2) $l_\infty \setminus \mathcal{C}$ ($= l_\infty \setminus \overline{\mathcal{C}}$ by (1)) is dense

Pf. Let $B_\varepsilon^\infty(x)$ be a ball in l_∞,

we need to show that $B_\varepsilon^\infty(x) \cap (l_\infty \setminus \mathcal{C}) \neq \phi$

If $x \in l_\infty \setminus \mathcal{C}$, we are done.

If $x \in \mathcal{C}$, then $x = \{x_n\}$ is convergent

Let $L = \lim_n x_n$

Then $\exists\, n_0 > 0$ s.t. $|x_n - L| < \frac{\varepsilon}{3}$, $\forall n \geq n_0$.

Define $y = \{y_n\} \in l_\infty$ by

$$y_n = \begin{cases} x_n, & \text{if } n < n_0 \\ L + \frac{\varepsilon}{3}, & \text{if } n \geq n_0 \ \& \ n \text{ odd} \\ L - \frac{\varepsilon}{3}, & \text{if } n \geq n_0 \ \& \ n \text{ even}. \end{cases}$$

Then $|x_n - y_n| = 0$ if $n < n_0$ and

$$|x_n - y_n| \leq |x_n - L| + |L - y_n|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} < \varepsilon, \quad \forall n \geq n_0$$

$\Rightarrow$ $d_\infty(x, y) \leq \frac{2\varepsilon}{3} < \varepsilon$ $\Rightarrow$ $y \in B_\varepsilon^\infty(x)$

However $\limsup y_n = L + \frac{\varepsilon}{3} > L - \frac{\varepsilon}{3} = \liminf y_n$.

$\therefore$ $y \in l_\infty \setminus \mathcal{C}$. $\Rightarrow$ $B_\varepsilon^\infty(x) \cap (l_\infty \setminus \mathcal{C}) \neq \phi$ #

Def: • A set in a metric space is called of first catogroy (or meager) if it can be expressed as a countable union of nowhere dence sets.

• A set is of second category if it is not of first categroy.

• A set is called residual if its complement is of first category.

Prop 4.8 Let (X, d) be a metric space.

(a) Every subset of a set of 1^{st} category is of 1^{st} category.

(b) The union of countable many sets of 1^{st} category is of 1^{st} category

(c) If (X, d) has no isolated point, then every countable subset of X is of 1^{st} category.