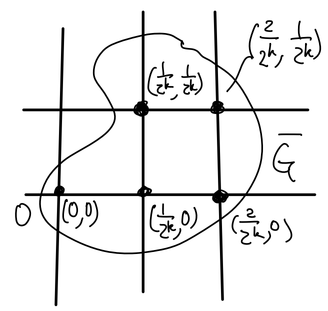
Thm 4.2 (Ascoli's Theorem)
Suppose that G is a bounded nonempty open set in $\mathbb{R}^m$. Then a set $\mathcal{E} \subset C(\overline{G})$ $(= C_b(\overline{G}))$ is precompact if $\mathcal{E}$ is bounded (in supnorm) and equicontinuous.

Pf: Define $E = \bigcup_{k=0}^{\infty} E_k$, where

$$E_k = \left\{ x = \frac{1}{2^k}\begin{pmatrix} l_1 \\ \vdots \\ l_m \end{pmatrix} \in \overline{G} : l_i \in \mathbb{Z},\ i=1,\cdots,m \right\}.$$



Then $\overline{G}$ closed and bounded
$\Rightarrow$ E_k is finite.
Hence $E = \bigcup_k E_k$ is countable.

Let $\{f_n\}$ be a sequence in $\mathcal{E}$. Then $\mathcal{E}$ bounded
$\Rightarrow$ $\exists M > 0$ such that $\|f_n\|_\infty \le M$, $\forall n$

i.e. $|f_n(x)| \le M$, $\forall n$ & $\forall x \in \overline{G}$
In particular, $\forall x \in E$,

$$|f_n(x)| \le M, \ \forall n.$$

i.e. If we arrange the points of E in a sequence $E = \{z_j\}_{j=1}^{\infty}$, then $\forall j \ge 1$,

$$\{f_n(z_j)\} \text{ is a bounded sequence.}$$

Hence one can apply Lemma 4.3 to find a subsequence

$\{g_n\}$ of $\{f_n\}$ (using the same notation "n" for the index)
such that $\forall\, x \in E$, $g_n(x)$ is convergent.

We claim that g_n is the required convergent subsequence of f_n in the metric space $(C(\overline{G}), d_\infty)$.

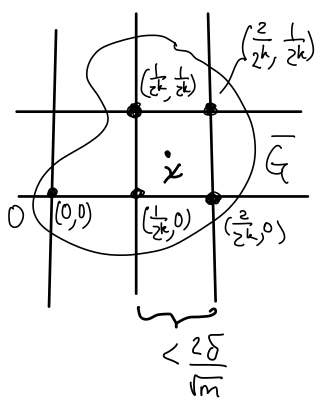
(Note that we only have pointwise convergence for countable many points at this moment.)

Since $(C(\overline{G}), d_\infty)$ is complete, we only need to show that $\{g_n\}$ is a Cauchy sequence in $(C(\overline{G}), d_\infty)$.
By equicontinuity of $\mathcal{E}$, ($\Rightarrow$ equicontinuity of $\{g_n\}$)

$\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that

$$|g_n(x) - g_n(y)| < \frac{\varepsilon}{3}, \quad \forall\, n \;\&\; \forall\, x, y \in \overline{G} \text{ with } |x-y| < \delta.$$

Note that if k satisfies $\frac{1}{2^k} < \frac{2\delta}{\sqrt{m}}$, then $\forall\, x \in \overline{G}$, $\exists\, z_j \in E_k$ such that $|x - z_j| < \delta$. (See figure)



and hence $|g_n(x) - g_n(z_j)| < \frac{\varepsilon}{3}$.

Therefore,

$$\begin{aligned} |g_n(x) - g_m(x)| &\le |g_n(x) - g_n(z_j)| + |g_n(z_j) - g_m(z_j)| \\ &\quad + |g_m(z_j) - g_m(x)| \\ &< \frac{2\varepsilon}{3} + |g_n(z_j) - g_m(z_j)|. \end{aligned}$$

Since $\{g_n(z_j)\}$ is convergent, $\exists\ n_0 = n_0(z_j) \geq 0$ s.t.

$$|g_n(z_j) - g_m(z_j)| < \frac{\varepsilon}{3}, \quad \forall\, n, m \geq n_0(z_j).$$

$\Rightarrow\ |g_n(x) - g_m(x)| < \varepsilon, \ \forall\, n, m \geq n_0(z_j).$ (z_j depends on x)

Now take $N_0 = \max\limits_{z_j \in E_k} n_0(z_j) \geq 0$, (finite set)

then $\forall\, x \in \overline{G}$, we have

$$|g_n(x) - g_m(x)| < \varepsilon, \quad \forall\, n, m \geq N_0.$$

i.e. $\|g_n - g_m\|_\infty < \varepsilon, \ \forall\, n, m \geq N_0.$

This completes the proof of the Theorem. #

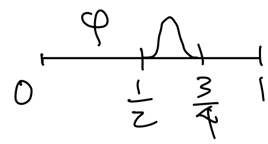
Remarks

(1) Ascoli's Theorem remains valid for bounded and equicontinuous subsets of $C(G)$. (i.e. No need to take closure.)
It is because "equicontinuous" $\Rightarrow$ "uniform continuous on G", and then can be extended to uniform continuous on $\overline{G}$.
(Details omitted.)

(2) However, <u>boundedness</u> of the domain $\overline{G}$ cannot be removed:

Eg 4.3 Let $\overline{G} = [0, \infty) \subset \mathbb{R}$.

Take a $\varphi \in C^1[0,1]$ such that



$$\varphi \not\equiv 0 \quad \text{and} \quad \varphi(x) = 0 \text{ on } [0,1] \setminus [\tfrac{1}{2}, \tfrac{3}{4}]$$

and define
$$f_n(x) = \begin{cases} \varphi(x-n), & \text{if } x \in [n, n+1] \\ 0, & \text{otherwise.} \end{cases}$$

Then one can easily check that

$$f_n \in C(\overline{G})$$ (in fact $f_n \in C^1(\overline{G})$)

and $\|f_n\|_{\infty, \overline{G}} = \|\varphi\|_{\infty, [0,1]} > 0$ (and a fixed constant)

$\therefore$ $\mathcal{E} = \{f_n\}$ is <u>bounded</u> subset in $C(\overline{G})$.

By Chain rule, $\left\|\frac{df_n}{dx}\right\|_{\infty, \overline{G}} = \left\|\frac{d\varphi}{dx}\right\|_{\infty, [0,1]}$ (>0.) indep. of n.

Hence Prop 4.1 implies that $\mathcal{E} = \{f_n\}$ is also <u>equicontinuous</u>.

Suppose $\exists$ subsequence $\{f_{n_j}\}$ of $\{f_n\}$ converges to some $f \in C(\overline{G})$ in d_∞.

i.e. $f_{n_j} \to f$ uniformly on $\overline{G}$

$\Rightarrow$ pointwise convergence $f_{n_j}(x) \to f(x)$, $\forall x \in \overline{G}$.

However, for fixed x, $f_n(x) = 0$, $\forall n \geq x$, we must have

$$\lim_{j \to +\infty} f_{n_j}(x) = 0. \quad \therefore \quad f(x) = 0, \ \forall x \in \overline{G}.$$

This is a contradiction, since

$$0 < \|\varphi\|_{\infty,[0,1]} = \|f_{n_j}\|_{\infty,\overline{G}} = \|f_{n_j} - f\|_{\infty,\overline{G}} \to 0 .$$

$\therefore$ $\mathcal{E}$ is <u>not precompact</u>.

Hence Ascoli's Theorem doesn't hold. #

Converse to Ascoli's Theorem:

Thm 4.4 (<u>Arzela's Theorem</u>) Suppose that G is a bounded nonempty open set in $\mathbb{R}^m$. Then every precompact set in $C(\overline{G})$ must be bounded and equicontinuous.

<u>Pf</u>: Let $\mathcal{E} \subset C(\overline{G})$ be precompact.

If $\mathcal{E}$ is unbounded, then $\exists f_n \in \mathcal{E} \subset C(\overline{G})$ such that

$$\lim_{n\to+\infty} \|f_n\|_\infty = \infty .$$

Then this subset $\{f_n\}$ of $\mathcal{E}$ cannot contain any convergent subsequence. This contradicts the precompactness. Hence $\mathcal{E}$ must be bounded.

Now suppose on the contrary that $\mathcal{E}$ is precompact, bounded but not equicontinuous.

Then $\exists\, \varepsilon_0 > 0$ such that $\forall\, \delta > 0$
$\exists\, x, y \in \overline{G}$ and $f \in \mathcal{E}$ satisfying

$$|f(x) - f(y)| \geq \varepsilon_0 \quad \& \quad d(x,y) < \delta.$$

In particular, by choosing $\delta = \frac{1}{n} > 0$, for $n = 1, 2, \cdots$
$\exists\, x_n, y_n \in \overline{G}$ and $f_n \in \mathcal{E}$ satisfying

$$|f_n(x_n) - f_n(y_n)| \geq \varepsilon_0 \quad \& \quad d(x_n, y_n) < \frac{1}{n}.$$

By precompactness, $\exists$ convergent subseq. $\{f_{n_k}\}$ of $\{f_n\}$.
Suppose $f \in C(\overline{G})$ is the limit,
i.e.

$$d_\infty(f_{n_k}, f) \to 0, \quad \text{as } k \to +\infty.$$

(i.e. f_{n_k} converges uniformly to f on $\overline{G}$)

Since $\overline{G}$ is closed and bounded, the corresponding sequences of points $\{x_{n_k}\}$ ($\{y_{n_k}\}$) contains convergent subsequence.
Denotes the subseq. by $\{x_k\}$ and assume $x_k \to z \in \overline{G}$.

And also denote the corresponding subseq. of $\{y_{n_k}\}$ by $\{y_k\}$, and the corresponding subseq. of $\{f_{n_k}\}$ by $\{g_k\}$.
Then

$$\begin{cases} g_k \to f & \text{in } (C(\overline{G}), d_\infty) \\ x_k \to z & \text{in } \overline{G} \end{cases}$$

Since $d(x_n, y_n) < \frac{1}{n}$, we have $d(x_k, y_k) \to 0$ as $k \to \infty$
and hence $y_k \to z \in \overline{G}$ too.

Therefore, $\forall \varepsilon > 0$, $\exists\, k_0 \geq 0$ s.t.

$$\|g_k - f\|_\infty < \varepsilon, \quad \forall k \geq k_0.$$

and $\exists\, k_1 \geq 0$ s.t.

$$\begin{aligned} |f(x_k) - f(z)| &< \varepsilon \\ |f(y_k) - f(z)| &< \varepsilon \end{aligned} \qquad \forall k \geq k_1$$

Hence for $k \geq \max\{k_0, k_1\}$,

$$\begin{aligned} |g_k(x_k) - g_k(y_k)| &\leq |g_k(x_k) - f(x_k)| + |f(x_k) - f(y_k)| \\ &\quad + |f(y_k) - g_k(y_k)| \\ &< 2\varepsilon + |f(x_k) - f(y_k)| \\ &\leq 2\varepsilon + |f(x_k) - f(z)| + |f(z) - f(y_k)| \\ &< 4\varepsilon \end{aligned}$$

We've show that $\forall \varepsilon > 0$, $\exists\, n_0 = n_{\max\{k_0, k_1\}} \geq 0$ such that

$$|f_{n_k}(x_{n_k}) - f_{n_k}(y_{n_k})| < 4\varepsilon, \quad \forall n_k \geq n_0$$

Taking $\varepsilon = \frac{\varepsilon_0}{4}$, we have a contradiction.

$\therefore\ \mathcal{E}$ is equicontinuous. #

Application to Ordinary Differential Equations

Consider (IVP) $\begin{cases} \frac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases}$

with f continuous (only, not necessary Lipschitz) on

$$R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b].$$

Of course, we cannot expect uniqueness result, but short time existence can be proved.

Idea of proof:

(1) Weierstrass Approximation Theorem (on $\mathbb{R}^2$)

$\Rightarrow \exists \{p_n\}$ sequence of polynomials s.t.

$$d_\infty(p_n, f) \to 0 \qquad (\text{in } C(R))$$

(2) Note that $\forall$ p_n satisfies Lipschitz condition (uniform in t).

By Picard-Lindelöf Theorem,

$$\exists\ a_n' > 0 \text{ with } a_n' < \min\{a, \frac{b}{M_n}, \frac{1}{L_n}\},$$

where $M_n = \|p_n\|_{\infty, R}$

L_n = Lipschitz constant of p_n on R,

s.t. $\exists$ unique solution $x_n \in C^1[t_0 - a_n', t_0 + a_n']$ to the

approximated (IVP) $\begin{cases} \frac{dx_n}{dt} = p_n(t, x_n) & \forall t \in [t_0 - a_n', t_0 + a_n'] \\ x_n(t_0) = x_0 \end{cases}$

(3) Then try to apply Ascoli's Theorem to $\{x_n\}$ and find a convergent subsequence $x_{n_k} \to x$ for some function $x(t)$.
And hope that x is the required solution.

<u>Issue</u>: Since f is not assumed to satisfy the Lipschitz condition, one cannot expect $\{L_n\}$ is bounded
(In fact, it is unbounded. Otherwise f satisfies Lip condition)
Then $\min\{a, \frac{b}{M_n}, \frac{1}{L_n}\} \to 0 \Rightarrow a_n' \to 0$.

We will not have an "interval" for the existence of the solution.
(On the other hand, as $p_n \to f$ in $(C(R), d_\infty)$, we have $M_n \le M$ for some $M > 0$.)

Therefore, to implement our plan, we need to improve the Picard-Lindelöf Theorem to

<u>Prop 4.5</u> Under the setting of Picard-Lindelöf Theorem, $\exists$ unique solution $x(t)$ on the interval $[t_0 - a', t_0 + a']$ with $x(t) \in [x_0 - b, x_0 + b]$, where a' is any number satisfying

$$0 < a' < a^* = \min\{a, \frac{b}{M}\}.$$

Clearly, this implies $\exists$ unique solution on the open interval $(t_0 - a^*, t_0 + a^*)$.

<u>Pf</u>: Omitted

Thm 4.6 (Cauchy-Peano Theorem)

Consider

$$(\text{IVP}) \begin{cases} \frac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

where f is continuous on $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$.

There exists $a' \in (0, a)$ and a C^1-function

$$x : [t_0 - a', t_0 + a'] \longrightarrow [x_0 - b, x_0 + b]$$

solving the (IVP).

Pf: As in the "Idea of Proof",

$\exists$ sequence of polynomials $\{p_n\}$ s.t.

$$p_n \to f \text{ in } (C(R), d_\infty).$$

This implies $M_n = \|p_n\|_{\infty, R} \to M$, where $M = \|f\|_{\infty, R}$,

and p_n satisfies the Lipschitz condition.

(we don't need to worry about the Lip. constants by Prop 5.4)

By Prop 4.5, $\exists$ unique solution x_n defined on $I_n = (t_0 - a_n, t + a_n)$, where $a_n = \min\{a, \frac{b}{M_n}\}$, for the (IVP)

$$\begin{cases} \frac{dx_n}{dt} = p_n(t, x_n) \\ x_n(t_0) = x_0 \end{cases}, \quad t \in I_n.$$

with $x_n(t) \in [x_0 - b, x_0 + b]$.

As $a_n = \min\{a, \frac{b}{M_n}\} \to \min\{a, \frac{b}{M}\} = a^*$, we have for any fixed $a' < a^*$ $(a'>0)$, $\exists\, n_0 > 0$ such that for $n \geq n_0$, $[t_0 - a', t_0 + a'] \subset J_n = (t_0 - a_n, t_0 + a_n)$.

Hence $\forall\, n \geq n_0$, x_n is defined on $[t_0 - a', t_0 + a']$.

Claim 1: $\{x_n\} \subset C[t_0 - a', t_0 + a']$ is equicontinuous.

In fact, (IVP) $\Rightarrow$ $\left|\frac{dx_n}{dt}\right| = |f_n(t, x_n)| \leq M_n \quad \forall t$

Since $M_n \to M$, $\left\|\frac{dx_n}{dt}\right\|_\infty$ is uniformly bounded.

By Prop 4.1, $\{x_n\}$ is equicontinuous.

Claim 2: $\{x_n\}$ is bounded in $C[t_0 - a', t_0 + a']$

In fact, (IVP) $\Rightarrow$ $x_n(t) = x_0 + \int_{t_0}^{t} f_n(s, x_n(s))\,ds$, $\forall t \in [t_0 - a', t_0 + a']$

$\therefore$ $|x_n(t)| \leq |x_0| + a' \sup_s |f_n(s, x_n(s))| \leq |x_0| + a' M_n$

$\Rightarrow$ $\|x_n\|_{\infty, [t_0 - a', t_0 + a']}$ is uniformly bounded.

$\therefore$ $\{x_n\}$ is a bounded set in $C[t_0 - a', t_0 + a']$.

Then Claims 1 & 2 allow us to apply Ascoli's Theorem to conclude that $\exists$ a subsequence x_{n_j} in $C[t_0 - a', t_0 + a']$ converges uniformly to a cts. function x on $[t_0 - a', t_0 + a']$.

<u>Claim 3</u>: x solves (IVP) $\begin{cases} \frac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases}$.

<u>Proof of Claim 3</u>: We only need to show that

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds.$$

Note that x_{n_j} satisfies

$$x_{n_j}(t) = x_0 + \int_{t_0}^{t} p_{n_j}(s, x_{n_j}(s))\,ds.$$

Clearly $x_{n_j}(t) \to x(t)$ as $j \to +\infty$. We only need to show that

$$\lim_{j\to\infty} \int_{t_0}^{t} p_{n_j}(s, x_{n_j}(s))\,ds = \int_{t_0}^{t} f(s, x(s))\,ds.$$

Since $f \in C(R)$ & R is closed & bounded in $\mathbb{R}^2$, f is uniformly continuous on R.

Therefore, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$\forall (s_1, x_1), (s_2, x_2) \in R$ with $|s_1 - s_2| < \delta$ and $|x_1 - x_2| < \delta$,

we have $\quad |f(s_2, x_2) - f(s_1, x_1)| < \varepsilon$.

On the other hand, $\|p_n - f\|_{\infty, R} \to 0$

$\Rightarrow \exists\, n_0 > 0$ s.t. $\quad |p_n(s,x) - f(s,x)| < \varepsilon$, $\forall (s,x) \in R$.

Therefore, for j sufficiently large such that

$$n_j \geq n_0 \quad \& \quad \|x_{n_j} - x\|_\infty < \delta,$$

we have

$$\left| \int_{t_0}^{t} f_{n_j}(s, x_{n_j}(s))\,ds - \int_{t_0}^{t} f(s, x(s))\,ds \right|$$

$$\leq \left| \int_{t_0}^{t} f_{n_j}(s, x_{n_j}(s))\,ds - \int_{t_0}^{t} f(s, x_{n_j}(s))\,ds \right| + \left| \int_{t_0}^{t} f(s, x_{n_j}(s))\,ds - \int_{t_0}^{t} f(s, x(s))\,ds \right|$$

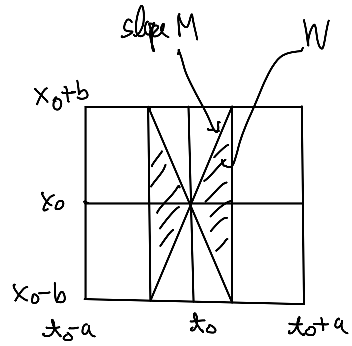
$$\leq \int_{t_0}^{t} \left| f_{n_j}(s, x_{n_j}(s)) - f(s, x_{n_j}(s)) \right| ds + \int_{t_0}^{t} \left| f(s, x_{n_j}(s)) - f(s, x(s)) \right| ds$$

$$\leq \varepsilon \cdot a' + \varepsilon \cdot a' = 2\varepsilon a',$$

This shows that $\int_{t_0}^{t} f_{n_j}(s, x_{n_j}(s))\,ds \to \int_{t_0}^{t} f(s, x(s))\,ds$ as $j \to +\infty$.

This completes the proof of Claim 3 and hence the theorem. ⌗

Another approach to Cauchy-Peano Theorem using Ascoli's Theorem
(Piecewise Linear Approximation)



Let $R = [t_0-a, t_0+a] \times [x_0-b, x_0+b]$

$M = \sup_R |f(t,x)|$ as before.

(May assume $M \geq 1$ as we only need an upper bd)

Define $W = \{(t,x) \in R : |x-x_0| \leq M|t-t_0|\}$

By symmetry,

proj(W) onto t-axis is $[t_0-a', t_0+a']$ for some $a' \in (0,a]$.

Note that $f \in C(R) \Rightarrow f \in C(W)$

$\Rightarrow f$ is uniformly continuous on W (since W is closed & bounded)

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ such that

$\forall (t_1,x_1), (t_2,x_2) \in W$ with

$|t_1-t_2| < \delta$ and $|x_1-x_2| < \delta$

we have

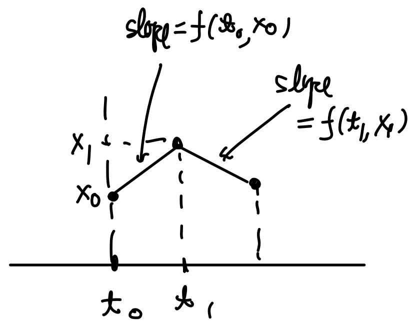
$$|f(t_2,x_2) - f(t_1,x_1)| < \varepsilon.$$

On the (half) interval $[t_0, t_0+a']$, choose

$$t_0 < t_1 < t_2 < \cdots < t_k = t_0+a'$$

with $$|t_i - t_{i-1}| < \frac{\delta}{M} \quad \text{for } i = 1, \dots k$$

Define a function $k_\varepsilon(t)$ on $[t_0, t_0+a']$



(1) $k_\varepsilon(t_0) = x_0$,

(2) $k_\varepsilon\big|_{[t_{i-1}, t_i]}$ is linear with slope $f(t_{i-1}, x_{i-1})$

where x_i can be determined succesively by :

(i) x_1 detemuied by $k_\varepsilon\big|_{[t_0, t_1]}$ is linear, its graph passing through (t_0, x_0) and with slope $f(t_0, x_0)$.

(ii) Note that $|f(t_0,x_0)| \le M$, $|x_1 - x_0| \le M|t_1 - t_0|$.

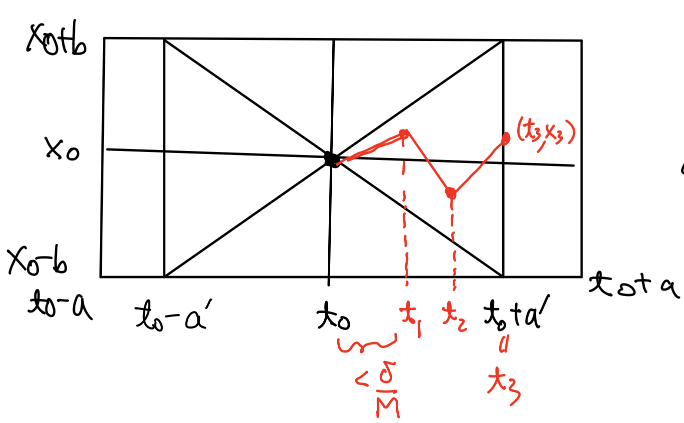
$\therefore$ $(t_1, x_1) \in W \subset R$ and hence $f(t_1, x_1)$ well-defined.

(iii) then x_2 detemined by $k_\varepsilon\big|_{[t_1, t_2]}$ is linear, its graph passing through (t_1, x_1) and with slope $f(t_1, x_1)$.

(iv) Similarly, $|f(t_0,x_0)|$ & $|f(t_1,x_1)| \le M$, we have

$|x_2 - x_0| \le M|t_2 - t_0|$

$\therefore$ $(t_2, x_2) \in W \subset R$ and $f(t_2, x_2)$ well-defined.



graph of $k_\varepsilon(t)$

And so on, the function $k_\varepsilon(t)$ is defined on $[t_0, t_0+a']$

Note that

(1) k_ε is piecewise linear,

(2) $|k_\varepsilon(t) - k_\varepsilon(s)| \leq M\,|t-s|$, $\forall\, t,s \in [t_0, t_0+a']$

(By slopes $|f(t_i,x_i)| \leq M$ on each subinterval.)

$\therefore \{k_\varepsilon\}_\varepsilon$ is equicontinuous (as subset of $C[t_0, t_0+a']$.)

(3) $\{k_\varepsilon\}$ is also uniformly bounded $[t_0, t_0+a']$.

In fact, W is convex and the ends points (t_i, x_i) with $x_i = k_\varepsilon(t_i)$ belongs to W, we have $(t, k_\varepsilon(t)) \in W$ by piecewise linearity. As $W \subset R$, $|k_\varepsilon(t) - x_0| \leq b$

and hence $|k_\varepsilon(t)| \leq |x_0| + b$, $\forall\, t \in [t_0, t_0+a']$ and $\forall\, \varepsilon > 0$.

Hence Ascoli's Theorem implies that $\{k_\varepsilon\}$ is precompact.
In particular, the sequence $\{k_{\frac{1}{n}}\}_{n=1}^{\infty}$ has a convergent subsequence $\{k_{\frac{1}{n_\ell}}\}$ in $C[t_0, t_0+a']$ with

$$k_{\frac{1}{n_\ell}}(t) \to k(t) \in C[t_0, t_0+a'], \text{ as } \ell \to +\infty.$$

To show $k(t)$ satisfies the differential equation, we first show that k_ε is an approximated solution (including $\varepsilon = \frac{1}{n_\ell} > 0$)

For this $\varepsilon > 0$, let $\delta > 0$ be the corresponding quatity for uniform continuity of f, and t_i as in the construction of $k_\varepsilon(x)$.

Consider $t \in [t_0, t_0 + a']$ and $t \neq t_i$, $i = 0, 1, \cdots, k-1$

Then $\exists\, j = 1, 2, \cdots k$ such that $t_{j-1} < t < t_j$.

Using $|t - t_{j-1}| < |t_j - t_{j-1}| < \frac{\delta}{M}$, we have

$$|k_\varepsilon(t) - k_\varepsilon(t_{j-1})| \leq M|t - t_{j-1}| < \delta,$$

Hence
$$\left| f(t_{j-1}, k_\varepsilon(t_{j-1})) - f(t, k_\varepsilon(t)) \right| < \varepsilon$$

Since k_ε is piecewise linear,

$$k_\varepsilon'(t) = f(t_{j-1}, k_\varepsilon(t_{j-1})) \qquad \text{(by our construction)}$$

Hence

$$|k_\varepsilon'(t) - f(t, k_\varepsilon(t))| < \varepsilon, \quad \forall t \in [t_0, t_0 + a'] \setminus \{t_0, t_1, \cdots, t_k\}.$$

As $k_\varepsilon(t_0) = x_0$, $k_\varepsilon(t)$ is an approximated solution to

$$(\text{IVP}) \quad \begin{cases} \dfrac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases} \quad \text{on } [t_0, t_0 + a']$$

in the sense that $\begin{cases} \dfrac{dk_\varepsilon}{dt} = f(t, k_\varepsilon) + \text{remainder} \\ x(t_0) = x_0 \end{cases}$ (except finitely many points)

with $\|\text{remainder}\|_\infty < \varepsilon$.

Integrating the ODE, we have

$$\Rightarrow \quad k_\varepsilon(t) = k_\varepsilon(t_0) + \sum_{i=1}^{j-1} \int_{t_{i-1}}^{t_i} k_\varepsilon'(s)\,ds + \int_{t_{j-1}}^{t} k_\varepsilon'(s)\,ds$$

$$= x_0 + \int_{t_0}^{t} k_\varepsilon'(s)\,ds$$

$$\Rightarrow \left| k_\varepsilon(t) - x_0 - \int_{t_0}^{t} f(s, k_\varepsilon(s))\,ds \right| \le \int_{t_0}^{t} |k_\varepsilon'(s) - f(s, k_\varepsilon(s))|\,ds < \varepsilon a'.$$

In particular, if we denote $g_\ell = k_{\frac{1}{n_\ell}}$, (ie $\varepsilon = \frac{1}{n_\ell} \to 0$),

then

$$\left| g_\ell(t) - x_0 - \int_{t_0}^{t} f(s, g_\ell(s))\,ds \right| \le \frac{a'}{n_\ell}, \quad \forall \ell = 1, 2, 3, \cdots$$

Hence

$$\left| k(t) - x_0 - \int_{t_0}^{t} f(s, k(s))\,ds \right|$$

$$\le \left| k(t) - x_0 - \int_{t_0}^{t} f(s, k(s))\,ds - g_\ell(t) + x_0 + \int_{t_0}^{t} f(s, g_\ell(s))\,ds \right|$$

$$+ \left| g_\ell(t) - x_0 - \int_{t_0}^{t} f(s, g_\ell(s))\,ds \right|$$

$$\le \| k - g_\ell \|_\infty + \int_{t_0}^{t} |f(s, g_\ell(s)) - f(s, k(s))|\,ds + \frac{a'}{n_\ell}.$$

Since $\|g_\ell - k\|_\infty \to 0$ and f is uniform continuity,

$$\int_{t_0}^{t} |f(s, g_\ell(s)) - f(s, k(s))|\,ds \to 0 \quad \text{as } j \to +\infty.$$

Therefore by letting $\ell \to +\infty$, we have

$$k(t) = x_0 + \int_{t_0}^{t} f(s, k(s))\,ds, \quad \forall t \in [t_0, t_0 + a'].$$

$$\Rightarrow \begin{cases} \dfrac{dk}{dt} = f(t, k(t)) & \forall t \in [t_0, t_0 + a'] \\ k(t_0) = x_0. \end{cases}$$

Similarly argument $\Rightarrow$ $\exists\ \tilde{k}$ on $t \in [t_0 - a', t_0]$

satisfying
$$\begin{cases} \dfrac{d\tilde{k}}{dt} = f(t, \tilde{k}(t)) & \forall\, t \in [t_0 - a', t_0] \\ \tilde{k}(t_0) = x_0 . \end{cases}$$

Note that by construction
$$\frac{dk}{dt}(t_0) = f(t_0, x_0) = \frac{d\tilde{k}}{dt}(t_0) .$$

Hence
$$x(t) = \begin{cases} k(t), & t \in [t_0, t_0 + a'] \\ \tilde{k}(t), & t \in [t_0 - a', t_0] \end{cases}$$

is $C^1[t_0 - a', t_0 + a']$ and solve the (IVP). ※

<u>Remarks</u> (i) This proof doesn't need the Picard-Lindelöf Theorem.

(ii) The spirit of this proof is more in line with solving the (IVP) numerically.

(iii) The 1^{st} proof solve "approximated problems";
the 2^{nd} proof solve the (original) problem "approximately".