Eg 4.1 (Equicontinuous, but unbounded)

Let $X = [-1,1]$ and consider

$$\mathcal{E} = \{ x \in C[-1,1] : x'(t) = t,\ t \in [-1,1] \}.$$

$\forall\ x \in \mathcal{E}$, $|x(t) - x(s)| \le \|x'\|_\infty |t-s| \le |t-s|$.

$\Rightarrow$ $\mathcal{E}$ is equicontinuous (as above).

But $\mathcal{E}$ is unbounded:

$$x_n(t) = \frac{t^2}{2} + n \in \mathcal{E} \text{ has}$$
$$\|x_n\|_\infty = \frac{1}{2} + n \to +\infty \text{ as } n \to +\infty.$$

(Clearly, $\{x_n\}$ has no convergent subsequence.)

eg 4.5 (Closed & Bounded, but not Equicontinuous)

Let $\mathcal{B} = \{ f \in C[0,1] : |f(x)| \le 1,\ \forall x \in [0,1] \}$ $\left(= B_1^\infty(0) \right)$

Then $\mathcal{B}$ is closed and bounded.

To show that $\mathcal{B}$ is not equicontinuous, we only need to find a subset of $\mathcal{B}$ which is not equicontinuous.

Let $\{ f_n(x) = \sin nx \}_{n=1}^\infty \subset \mathcal{B}$.

Suppose on the contrary that

$$\{ f_n(x) = \sin nx \}_{n=1}^\infty \text{ is equicontinuous.}$$

Then for $\varepsilon = \frac{1}{2}$, $\exists\ \delta > 0$ such that

$\forall n \geq 1$, & $x, y \in [0,1]$ with $|x-y| < \delta$, we have

$$|\sin nx - \sin ny| < \frac{1}{2}.$$

However, for any $\delta > 0$, if $n > \max\left\{\frac{\pi}{2\delta}, \frac{\pi}{2}\right\}$,

we have $x = 0$ & $y = \frac{\pi}{2n} \in [0,1]$ with $|x-y| < \delta$

and $|\sin n\cdot 0 - \sin n\cdot\frac{\pi}{2n}| = |0-1| = 1 > \frac{1}{2}$

which is a contradiction.

$\therefore$ $\{\sin nx\}_{n=1}^{\infty}$ is not equicontinuous. ※

<u>Lemma 4.3</u> Let $A = \{z_j\}_{j=1}^{\infty}$ be a <u>countable set</u> and $f_n : A \to \mathbb{R}$, $n = 1, 2, \cdots$, be a sequence of functions defined on A. Suppose that for each $z_j \in A$,

$$\{f_n(z_j)\}_{n=1}^{\infty} \text{ is a } \underline{\text{bounded sequence}} \text{ in } \mathbb{R}.$$

Then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $\forall z_j \in A$,

$$\{f_{n_k}(z_j)\} \text{ is convergent.}$$

<u>Pf</u>: Since $f_n(z_1)$ is a bounded sequence (in $\mathbb{R}$), $\exists$ a subsequence f_n^1 such that

$$f_n^1(z_1) \text{ is convergent.}$$

(Note that we have used the same index n to denote the subsequence f_{n_k}. The superscript 1 is to denote that it is convergent when evaluated at z_1.)

For this subsequence f_n^1 (of original f_n),

$$f_n^1(z_2) \text{ is bounded } \left(\text{since } \{f_n^1(z_2)\} \subset \{f_n(z_2)\}\right).$$

Hence $\exists$ a subsequence $\{f_n^2\}$ of $\{f_n^1\}$ such that

$$\{f_n^2(z_2)\} \quad \text{is convergent.}$$

Note that since $\{f_n^1\}$ is a subseq. of $\{f_n\}$, $\{f_n^2\}$ is also a subsequence of $\{f_n\}$.

Also, $\{f_n^2(z_1)\}$ is a subseq. of the convergent subsequence $\{f_n^1(z_1)\}$,

$\therefore$ $\{f_n^2(z_1)\}$ is also convergent.

Therefore, we've found a subseq. $\{f_n^2\}$ of $\{f_n\}$ such that

$$\{f_n^2(z_1)\} \text{ and}$$
$$\{f_n^2(z_2)\} \text{ are convergent.}$$

And $\{f_n^2\}$ is a subseq. of $\{f_n^1\}$.

Repeating the process, one can obtain sequences $\{f_n^j\}$ (with $f_n^0 = f_n$) such that

(i) $\{f_n^{j+1}\}$ is a subsequence of $\{f_n^j\}$, $\forall j = 0, 1, 3 \cdots$

(ii) $\{f_n^j(z_1)\}, \{f_n^j(z_2)\}, \cdots, \{f_n^j(z_j)\}$ are convergent $(j \geq 1)$

$$
\begin{array}{cccccccc|c}
 & & & & & & & & \text{convergent at} \\
f_1^1 & f_2^1 & f_3^1 & \cdots & f_n^1 & \cdots & & & z_1 \\
f_1^2 & f_2^2 & f_3^2 & \cdots & f_n^2 & \cdots & & & z_1, z_2 \\
f_1^3 & f_2^3 & f_3^3 & \cdots & f_n^3 & \cdots & & & z_1, z_2, z_3 \\
\vdots & \vdots & \vdots & & \vdots & & & & \vdots \\
f_1^n & f_2^n & f_3^n & \cdots & f_n^n & \cdots & & & z_1, z_2, \cdots, z_n \\
\vdots & \vdots & \vdots & & \vdots & & & &
\end{array}
$$

Define $g_n = f_n^n$, $\forall n \geq 1$. (the diagonal sequence),

then $\{g_n\}$ is a subsequence of $\{f_n\}$ and

for any fixed $j = 1, 2, \cdots$, $g_n(z_j) = f_n^n(z_j)$

As $n \to \infty$, $n \geq j$ for sufficiently large n.

Hence $\{f_n^n(z_j)\}$ is a subsequence of the convergent sequence

$\{f_n^j(z_j)\}$ for all sufficiently large n.

Therefore $\{g_n(z_j)\}$ is convergent.

This completes the proof of the Lemma. ※

(This method of finding g_n is called the Cantor's diagonal trick.)