eg 2.13 $X = C[a,b]$ with $d_\infty(f,g) = \|f-g\|_\infty = \sup\limits_{x\in[a,b]} |f-g|(x)$

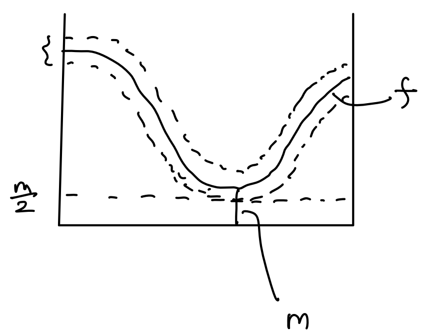
let $E = \{f \in C[a,b] : f(x) > 0,\ \forall x \in [a,b]\} \subset X$

$\forall f \in E$, f is positive, cts on the closed & bounded interval $[a,b]$, therefore $\exists\, m > 0$ s.t.

$$f(x) \geq m > 0, \quad \forall x \in [a,b].$$

Consider $B^\infty_{\frac{m}{2}}(f) = \{g \in C[a,b] : d_\infty(g,f) < \frac{m}{2}\}$

$\forall g \in B^\infty_{\frac{m}{2}}(f)$, we have $\forall x \in [a,b]$.



$$\begin{aligned} g(x) &= [g(x) - f(x)] + f(x) \\ &\geq f(x) - \|g-f\|_\infty \\ &> f(x) - \frac{m}{2} \geq m - \frac{m}{2} = \frac{m}{2} > 0 \end{aligned}$$

$\therefore$ $g \in E$ & hence $B^\infty_{\frac{m}{2}}(f) \subset E$.

$\therefore$ E is open in $(C[a,b], d_\infty)$.

Similarly, one can show that $\forall \alpha \in \mathbb{R}$

$$\{f \in C[a,b] : f(x) > \alpha,\ \forall x \in [a,b]\}$$
$$\{f \in C[a,b] : f(x) < \alpha,\ \forall x \in [a,b]\}$$

are open in $(C[a,b], d_\infty)$.

And
$$\{f \in C[a,b] : f(x) \geq \alpha, \ \forall x \in [a,b]\}$$
$$\{f \in C[a,b] : f(x) \leq \alpha, \ \forall x \in [a,b]\}$$
are closed in $(C[a,b], d_\infty)$ (Ex!)

(Caution: $C[a,b] \setminus \{f \in C[a,b] : f(x) \geq \alpha, \forall x \in [a,b]\}$
$\neq \{f \in [a,b] : f(x) < \alpha, \forall x \in [a,b]\}$)

eg 2.14: Let $X \neq \emptyset$ and d = discrete metric on X.

Then $\forall$ subset $E \subset X$,
$$B_{\frac{1}{2}}(x) = \{x\} \subset E, \quad \forall x \in E.$$
(HW4)

$\therefore$ E is open.

Therefore, any subset E of $(X, \text{discrete})$ is open, & hence any subset E of $(X, \text{discrete})$ is closed.

Together, any subset E of $(X, \text{discrete})$ is both open and closed.

In particular, any $\{x\} \subset (X, \text{discrete})$ is both open and closed.

Prop 2.6 Let (X, d) be a metric space. A sequence $\{x_n\}$ converges to x <u>if and only if</u> $\forall$ open set G containing x, $\exists n_0$ such that $x_n \in G$, $\forall n \geq n_0$.

Pf: $(\Rightarrow)$ Let G open & $x \in G$

$\Rightarrow \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset G$

As $x_n \to x$, for this $\varepsilon > 0$, $\exists n_0$ s.t.

$$d(x_n, x) < \varepsilon, \ \forall n \geq n_0$$

$$\Rightarrow \quad x_n \in B_\varepsilon(x) \subset G, \ \forall n \geq n_0$$

$(\Leftarrow)$ $\forall \varepsilon > 0$, $B_\varepsilon(x)$ is an open set containing x.

Therefore $\exists n_0$ s.t. $x_n \in B_\varepsilon(x)$, $\forall n \geq n_0$

$$\Rightarrow \quad d(x_n, x) < \varepsilon, \ \forall n \geq n_0$$

\#

Prop 2.7 Let (X, d) be a metric space. Then a set $A \subset X$ is closed <u>if and only if</u> whenever $\{x_n\} \subset A$ and $x_n \to x$ as $n \to \infty$ implies that $x \in A$.

Pf: ($\Rightarrow$) Suppose not. Then $x \notin A$

i.e. $x \in X \setminus A$ which is open (as A closed)

$\Rightarrow \exists \varepsilon > 0$, $B_\varepsilon(x) \subset X \setminus A$.

On the other hand $x_n \to x$, $\exists n_0$ s.t. $d(x_n, x) < \varepsilon$ $\forall n \geq n_0$

$\Rightarrow x_n \in B_\varepsilon(x) \subset X \setminus A$

$\Rightarrow x_n \notin A$ contradiction ※

($\Leftarrow$) Suppose not. Then A is not closed.

$\Leftrightarrow$ $X \setminus A$ is not open

$\exists x \in X \setminus A$ s.t. $B_\varepsilon(x) \not\subset X \setminus A$, $\forall \varepsilon > 0$.

In particular, $B_{\frac{1}{n}}(x) \cap A \neq \emptyset$, $\forall n = 1, 2, \ldots$

Pick $x_n \in B_{\frac{1}{n}}(x) \cap A$ for each n

Then $\{x_n\} \subset A$ & $d(x_n, x) < \frac{1}{n}$, $\forall n$

$\Rightarrow x_n \to x$ as $n \to \infty$.

Contradicting the assumption (as $x \in X \setminus A$.) ※

Prop 2.8 Let $f:(X,d)\to(Y,\rho)$ be a mapping between metric spaces.

(a) f is continuous at x
$\Leftrightarrow$ $\forall$ open set G (in Y) containing $f(x)$,
$f^{-1}(G)$ contains $B_\varepsilon(x)$ for some $\varepsilon>0$.

(b) f is continuous in X
$\Leftrightarrow$ $\forall$ open set G in Y, $f^{-1}(G)$ is <u>open</u> in X

Pf: (a) ($\Rightarrow$) Suppose not,
then $\exists$ open set G in Y containing $f(x)$
s.t. $f^{-1}(G)$ doesn't containing $B_\varepsilon(x)$, $\forall\varepsilon>0$.

i.e. $B_\varepsilon(x)\cap[X\setminus f^{-1}(G)]\neq\phi$, $\forall\varepsilon>0$.

In particular $B_{\frac{1}{n}}(x)\cap[X\setminus f^{-1}(G)]\neq\phi$, $\forall n$.

Pick $x_n\in B_{\frac{1}{n}}(x)\cap[X\setminus f^{-1}(G)]$, $\forall n$.

Then
$$\begin{cases} x_n\in B_{\frac{1}{n}}(x)\Rightarrow x_n\to x \text{ as } n\to\infty \\ x_n\in X\setminus f^{-1}(G)\Rightarrow f(x_n)\notin G,\ \forall n \end{cases}$$

By Prop 2.6, $f(x_n) \not\to f(x)$. Contradicting the assumption that f is cts. at x.

$(\Leftarrow)$ $\forall \varepsilon > 0$, $B_\varepsilon(f(x)) \subset Y$ is an open set containing $f(x)$. By assumption,

$$f^{-1}(B_\varepsilon(f(x))) \supset B_\delta(x) \text{ for some } \delta > 0$$

i.e $\quad f(y) \in B_\varepsilon(f(x)), \quad \forall y \in B_\delta(x)$

$$\Rightarrow \quad d(f(y), f(x)) < \varepsilon, \quad \forall\, d(y,x) < \delta.$$

$\therefore$ f is cts. at x.

(b) follows from (a). (Ex!) $\divideontimes$

<u>Note</u>: We also have:

f is cts in X

$\Longleftrightarrow$ $\forall$ closed set $F \subset Y$, $f^{-1}(F)$ is closed in X.

(Pf: Ex!)

Eg (i) Let $A \subset X$ & $A \neq \emptyset$.

Since $\rho_A(x) = d(x, A)$ is cts, in $\mathbb{R}$

$$G_r = \{x \in X : d(x, A) < r\} = \rho_A^{-1}(B_r(0))$$

is open in X.

(ii) <u>Claim</u>: If A is closed $(\neq \emptyset)$, then $A = \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$.

Hence any closed set is a <u>countable intersection</u> of open sets.

Pf: It is clear that $A \subset \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$ as $A \subset G_{\frac{1}{n}}, \forall n$.

Let $x \in \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$ then $x \in G_{\frac{1}{n}}, \forall n$

$\Rightarrow \quad d(x, A) < \frac{1}{n}, \quad \forall n$

$\Rightarrow \quad \exists\, x_n \in A$ s.t. $d(x, x_n) < \frac{1}{n}, \quad \forall n$

Hence $\{x_n\} \subset A$ is a seq in A s.t. $x_n \to x$.

Since A is closed, we have $x \in A$. (Prop 2.7)

$\therefore \quad A = \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$ ※

§2.4 Points in Metric Spaces

Def: Let E be a set in a metric space (X, d)

(1) A point $x \in X$ (not nec. in E) is called a <u>boundary point</u> of E if $\forall$ open set $G \subset X$ containing x, $G \cap E \neq \phi$ & $G \setminus E \neq \phi$

$$(G \cap (X \setminus E) \neq \phi)$$

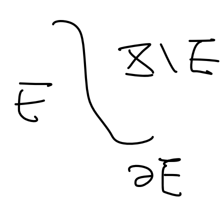
(2) The set of boundary points of E will be denoted by $\underline{\partial E}$ and is called the <u>boundary of E</u>.

(3) The <u>closure</u> of E, denoted by $\overline{E}$, is defined to be $\overline{E} = E \cup \partial E$.

<u>Note</u>:

(i) In (1), it suffices to check G of the form $B_\varepsilon(x)$ for all small $\varepsilon > 0$, or even $B_{\frac{1}{n}}(x)$, $\forall n \geq 1$ (See the proof of Prop 2.9(a)).

(ii) $\partial E = \partial(X \setminus E)$, $\forall E \subset X$.



eg: For $B_r(x) = \{y \in X : d(y,x) < r\}$ in $(\mathbb{R}^n, \text{standard})$

$$\partial B_r(x) = S_r(x) = \{y \in X : d(y,x) = r\} \quad \&$$

$$\overline{B_r}(x) = B_r(x) \cup \partial B_r(x) = \{y \in X : d(y,x) \le r\}$$

Further Notes

(i) $\partial \emptyset = \emptyset$ (Ex!)

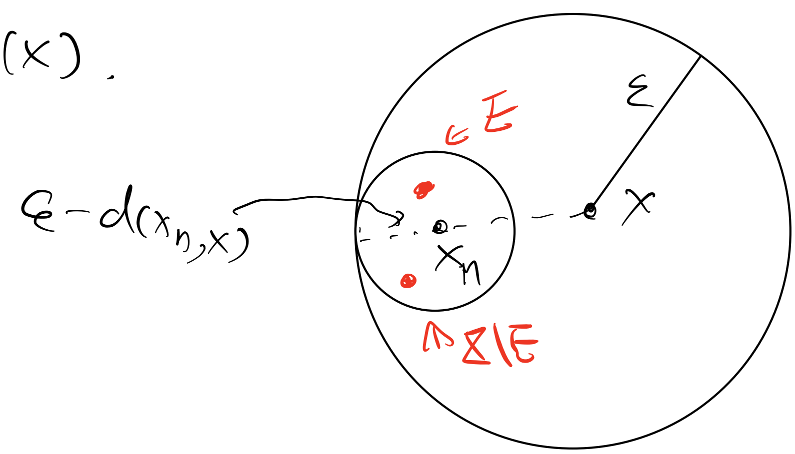
(ii) $\forall E \subset X$, ∂E is a closed set.

(iii) If E is closed, then $\overline{E} = E$.

Pf of (ii): Consider a seq $\{x_n\} \subset \partial E$ converging to some $x \in X$.

Then $\forall \varepsilon > 0$, $x_n \in B_\varepsilon(x)$ for $n \ge n_0$ (for some n_0)

$$\Rightarrow \quad B_{\varepsilon - d(x_n,x)}(x_n) \subset B_\varepsilon(x).$$



As $x_n \in \partial E$,

$$\begin{cases} B_{\varepsilon - d(x_n,x)}(x_n) \cap E \neq \emptyset \\ B_{\varepsilon - d(x_n,x)}(x_n) \setminus E \neq \emptyset \end{cases}$$

$$\Rightarrow \begin{cases} B_\varepsilon(x) \cap E \neq \emptyset \\ B_\varepsilon(x) \setminus E \neq \emptyset \end{cases} \xRightarrow{\text{(since } \varepsilon > 0 \text{ arbitrary)}} x \in \partial E.$$

Therefore ∂E is closed $\#$

Pf of (iii): Only need to show that

$\partial E \subset E$ if E is closed.

Let $x \in \partial E$, then by definition

$$B_{\frac{1}{n}}(x) \cap E \neq \phi \quad (\& \ B_{\frac{1}{n}}(x) \cap (\mathbb{X} \setminus E) \neq \phi)$$

$$\Rightarrow \quad \exists \ x_n \in B_{\frac{1}{n}}(x) \cap E.$$

$$\Rightarrow \quad d(x_n, x) < \frac{1}{n}, \quad \forall n$$

$\therefore \ x_n \to x$

Since E is closed, Prop 2.7 $\Rightarrow x \in E$.

Since $x \in \partial E$ is arbitrary, $\partial E \subset E$. #

Prop 2.9 Let $E \subset (\mathbb{X}, d)$. Then

(a) $x \in \overline{E} \Leftrightarrow B_r(x) \cap E \neq \emptyset, \ \forall r > 0$.

(b) $A \subset B \Rightarrow \overline{A} \subset \overline{B} \quad \forall A, B \subset (\mathbb{X}, d)$

(c) $\overline{E}$ is closed

(d) $\overline{E} = \cap \{C : C = \text{closed set}, \ C \supset E\}$.

(i.e. $\overline{E}$ is the smallest closed set containing E)

Pf (a) ($\Rightarrow$) $x \in \overline{E} \Rightarrow x \in E$ or $x \in \partial E$.

If $x \in E$, then $x \in B_r(x) \cap E$, $\forall r > 0$

$\Rightarrow B_r(x) \cap E \neq \phi$, $\forall r > 0$.

If $x \in \partial E$, then by definition of boundary point, $\forall$ open set G containing x, $G \cap E \neq \phi$

($\&$ $G \setminus E \neq \phi$)

Since $B_r(x)$ is open and $x \in B_r(x)$, $\forall r > 0$, we have $B_r(x) \cap E \neq \phi$, $\forall r > 0$.

($\Leftarrow$) If $x \in E$, we are done. ($x \in E \subset \overline{E}$)

If $x \notin E$, then for any open set G containing x, $x \in G \setminus E$. Hence $G \setminus E \neq \phi$.

To show that $G \cap E \neq \phi$, we choose $r_0 > 0$ s.t. $B_{r_0}(x) \subset G$ (it is possible since G is open).

Then by assumption, $B_{r_0}(x) \cap E \neq \phi$ and hence $G \cap E \left(\supset B_{r_0}(x) \cap E\right) \neq \phi$. #

(b) Let $x \in \overline{A}$.

By part (a), $B_r(x) \cap A \neq \phi$, $\forall r > 0$

Since $A \subset B$, $B_r(x) \cap B \neq \emptyset$, $\forall r > 0$

Part (a) again, $x \in \overline{B}$.

$\therefore \overline{A} \subset \overline{B}$. #

(c) Consider a seq $\{x_n\} \in \overline{E}$ such that $x_n \to x$ for some $x \in X$. We need to show that $x \in \overline{E}$

(Prop 2.7)

Suppose not, then $x \notin \overline{E}$.

Part (a) $\Rightarrow \exists\ \varepsilon_0 > 0$ such that $B_{\varepsilon_0}(x) \cap E = \phi$

For this $\varepsilon_0 > 0$, $\exists\ n_0 > 0$ such that $x_n \in B_{\varepsilon_0}(x)$ $\forall n \geq n_0$.

Then $B_{\varepsilon_0}(x) \cap E = \phi \Rightarrow x_n \in \partial E \setminus E$ for $n \geq n_0$.

In particular $\{x_n\}_{n=n_0}^{\infty}$ is a seq. in ∂E and $x_n \to x$. By Note (ii) above and prop 2.7, $x \in \partial E \subset \overline{E}$ which is a contradiction.

#

(d) By (c), $\overline{E}$ is closed & $\overline{E} \supset E$

$\therefore \overline{E} \in \{C : C = \text{closed set}, C \supset E\}$

$\Rightarrow \overline{E} \supset \cap \{C : C = \text{closed set}, C \supset E\}$

Conversely, let C be a closed set & $C \supset E$.

Then by (6) and (iii) of Further Notes above,

$$\overline{E} \subset \overline{C} = C$$

$$\Rightarrow \quad \overline{E} \subset \bigcap\{C : C = \text{closed set}, C \supset E\}$$ ∦

Def: Let E be a subset of a metric space (X, d).

(1) A point x is called an <u>interior point</u> of E if $\exists$ an open set G s.t. $x \in G$ & $G \subset E$.

(2) The set of all interior points of E is called the <u>interior of E</u>, denoted by E°.

<u>Notes</u>: (i) E° is open

(ii) $E^{\circ} = E \setminus \partial E$

(iii) $E^{\circ} = X \setminus \overline{(X \setminus E)}$

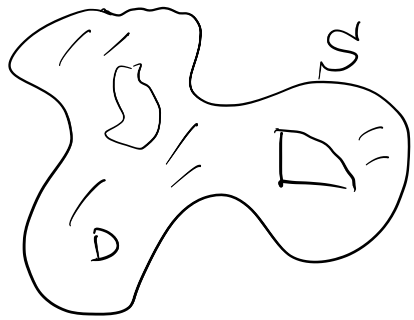
(iv) $E^{\circ} = \bigcup\{G : G = \text{open} \ \& \ G \subset E\}$

(Pf: Ex!)

<u>eg 2.18</u> $E = \mathbb{Q} \cap [0,1]$ in $(X = [0,1], d(x,y) = |x-y|)$

Then $E^{\circ} = \emptyset$ & $\overline{E} = [0,1]$, $\partial E = ?$

eg 2.19 Let D be a domain in $\mathbb{R}^2$ bounded by several cts. curves S.



Then $\partial D = S$

$\bar{D} = D \cup S = D \cup \partial D$

& $(\bar{D})^\circ = D$.

eg 2.20 : (i) $\overline{E \cup F} = \bar{E} \cup \bar{F}$ for $E, F \subset (X, d)$
(Ex!)

(ii) However $(E \cup F)^\circ \neq E^\circ \cup F^\circ$ in general.

Counterexample: $E = \mathbb{Q}$, $F = \mathbb{R} \setminus \mathbb{Q}$, $(X, d) = (\mathbb{R}, \text{standard})$

Then $E \cup F = \mathbb{R}$

$\Rightarrow (E \cup F)^\circ = \mathbb{R}$

However $E^\circ = F^\circ = \emptyset$

$\Rightarrow E^\circ \cup F^\circ = \emptyset \neq \mathbb{R} = (E \cup F)^\circ$.

(iii) We only have $E^\circ \cup F^\circ \subset (E \cup F)^\circ$ (Pf: Ex!)

eg 2.21 : $(X = C[0,1],\ d_\infty(f,g) = \|f - g\|_\infty)$

Let $S = \{f \in X : 1 < f(x) \leq 5,\ \forall x \in [0,1]\}$

(1) Claim: $\bar{S} = \{f \in X : 1 \leq f(x) \leq 5,\ \forall x \in [0,1]\}$.

Pf: Let $C' = \{ f \in X : 1 \le f(x) \le 5,\ \forall x \in [0,1] \} \supset S'$.

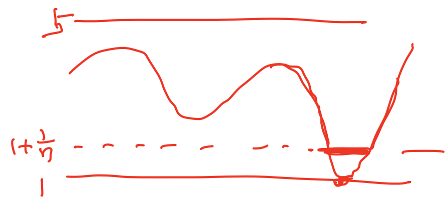
Then $C' = \{ 1 \le f(x) \} \cap \{ f(x) \le 5 \}$ closed in (X, d_∞) (Ex!)

$\therefore$ C' is closed.

$\therefore$ $\overline{S'} \subset C'$ (by Prop 2.9 (d))

Conversely, $\forall f \in C'$

$$f_n(x) = \max\{ f(x), 1+\tfrac{1}{n} \} \in X = C[0,1], \ \forall n.$$



Then $1 < 1+\frac{1}{n} \le f_n(x) \le 5$, $\forall n$

$\Rightarrow$ $f_n(x) \in S'$, $\forall n$

Note $d_\infty(f_n, f) = \max_{x \in [0,1]} |f_n - f|(x)$

$$\le 1 + \frac{1}{n} - 1 = \frac{1}{n} \to 0 \text{ as } n \to \infty$$

$\therefore$ $f \in \overline{S'}$ as $f_n \to f$.

Hence $C' \subset \overline{S'}$. ※

(2) <u>Claim</u>: $S'^{\circ} = \{ f \in X : 1 < f(x) < 5,\ \forall x \in [0,1] \}$.

(Pf: Ex!)