MATH3060 HW3 Due date: Oct 15, 2021 (at 12:00 noon)

1. (a) Show that $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ is a metric on $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.

   (b) Let (X, d) be a metric space. Show that

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

   is also a metric.

2. Let $X = C[a,b]$, $d_1(f,g) = \int_a^b |f-g|$, and $d_2(f,g) = \left(\int_a^b |f-g|^2\right)^{\frac{1}{2}}$ (for $f, g \in X$).

   (a) Is d_1 stronger than d_2?

   (b) Is d_2 stronger than d_1?

   (Hint: You may use Hölder's Inequality

$$\int_a^b |fg| \le \left(\int_a^b |f|^p\right)^{\frac{1}{p}} \left(\int_a^b |g|^q\right)^{\frac{1}{q}} \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1$$
)

3. Let $C^1[a,b] = \{f \in C[a,b] : f \text{ is continuous differentiable on } [a,b]\}$.

   Define $\forall f, g \in C^1[a,b]$

$$d(f,g) = \|f-g\|_\infty + \|f'-g'\|_\infty .$$

   Show that d is a metric on $C^1[a,b]$. Furthermore, Is $f_k(x) = \int_0^{1/k} \sin(ktx)\,dt$, $k = 1, 2, \cdots$ a convergence sequence in $(C^1[0,1], d)$?

4. Let $X = C[a,b]$ and $d_\infty(f,g) = \sup_{[a,b]} |f(x)-g(x)|$

(a) Show that d_∞ is a metric on X.

(b) Define $S : X \to X$ by

$$(Sf)(x) = \int_a^x f(t)\,dt, \quad \forall f \in X.$$

Show that S is continuous mapping (i.e. continuous at every $f \in X$) from the metric space (X, d_∞) to itself.

(End)