§2.2 Limits & Continuity

Def: A sequence $\{x_n\}$ in a metric space (X, d) is said to be <u>converge</u> to $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x) = 0$$

In this case, we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ in X.

Prop (<u>Uniqueness of limit</u>)

If $x_n \to x$ & $x_n \to y$ in a metric space, then $x = y$

(Pf: same as in $\mathbb{R}^n$ by using (M1))

<u>egs</u> (i) Convergence in $(\mathbb{R}^n, d_2)$ is the usual convergence in Adv. Calculus.

(ii) Convergence in $(C[a,b], d_\infty)$ is the <u>uniform</u> convergence of a seq. of functions in $C[a,b]$.

Def: Let d and ρ be 2 metrics defined on X.

(1) We call ρ is <u>stronger than</u> d or d is <u>weaker than</u> ρ, if $\exists\, C > 0$ s.t.

$$d(x,y) \le C \rho(x,y), \quad \forall x, y \in X$$

(2) They are <u>equivalent</u> if ρ is stronger and weaker than d.

i.e. $\exists\, C_1, C_2 > 0$ s.t.

$$d(x,y) \le C_1 \rho(x,y) \le C_2 d(x,y) \quad \forall x, y \in X.$$

(or $c_1 d(x,y) \le \rho(x,y) \le c_2 d(x,y)$)

Prop: (1) If ρ is stronger than d, then $\{x_n\}$ converges in (X,ρ) implies $\{x_n\}$ converges in (X,d), and have the same limit.

(2) If ρ is equivalent to d, then $\{x_n\}$ converges in (X,ρ) if and only if $\{x_n\}$ converges in (X,d).

(3) "equivalent" of metrics defined above is an equivalent relation.

(Pf: Easy ex!)

eg On $\mathbb{R}^n$,

$$\begin{cases} d_1(x,y) = \sum_i |x_i - y_i| \\ d_2(x,y) = \left(\sum_i |x_i-y_i|^2\right)^{1/2} \\ d_\infty(x,y) = \max_i |x_i - y_i| \end{cases}$$

Check:

(i) $d_2(x,y) \le \sqrt{n}\, d_\infty(x,y) \le \sqrt{n}\, d_2(x,y)$

(ii) $d_1(x,y) \le n\, d_\infty(x,y) \le n\, d_1(x,y)$

Therefore, d_1, d_2 & d_∞ are equivalent metrics on $\mathbb{R}^n$.

eg $X = C[a,b]$

$$\begin{cases} d_1(f,g) = \int_a^b |f-g| \\ d_\infty(f,g) = \max_{[a,b]} |f-g| \end{cases}$$

Then clearly

$$d_1(f,g) \le (b-a)\, d_\infty(f,g), \quad \forall f,g \in C[a,b]$$

$\therefore$ d_∞ is stronger than d_1.

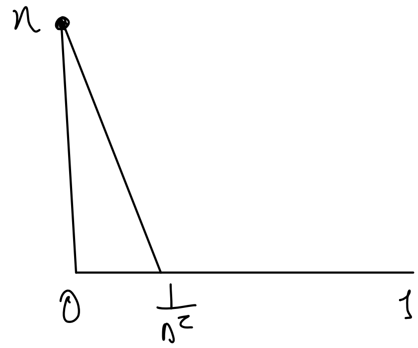
However, it is impossible to find $C>0$ s.t.

$$d_\infty(f,g) \le C\, d_1(f,g), \quad \forall f,g \in C[a,b]$$

Pf: Define f_n on $[a,b]=[0,1]$

$$f_n(x) = \begin{cases} -n^3x+n, & x\in[0,\frac{1}{n^2}] \\ 0, & x\in(\frac{1}{n^2},1]. \end{cases}$$

Then



$$d_1(f_n,0) = \int_0^1 |f_n| = \frac{1}{2n} \to 0 \text{ as } n\to\infty$$

$$d_\infty(f_n,0) = \max_{[0,1]} |f_n(x)| = n \to \infty \text{ as } n\to\infty$$

$\therefore$ $n = d_\infty(f_n,0) \le C\, d_1(f_n,0) = \frac{C}{2n}$, $\forall n$

which is impossible.

$\Rightarrow$ d_1 is not stronger than d_∞.

Therefore d_1 & d_∞ are not equivalent.

Def: Let $f:(X,d)\to(Y,\rho)$ be a mapping between two metric spaces, and $x\in X$. We call f is *continuous at x* if

$f(x_n)\to f(x)$ in (Y,ρ) whenever $x_n\to x$ in (X,d).

It is *continuous* on a set $E\subset X$ if it is continuous at every point of E.

Prop 2.2 Let $f:(X,d)\to(Y,\rho)$ be a mapping between 2 metric spaces, and $x_0\in X$. Then

f is continuous at x_0

$\Leftrightarrow$ $\begin{cases} \forall \varepsilon>0,\ \exists\delta>0 \text{ such that} \\ \rho(f(x),f(x_0))<\varepsilon, \ \forall x \text{ with } d(x,x_0)<\delta \end{cases}$

(Pf: Ex!)

Prop 2.3: Let $f: (X, d) \to (Y, \rho)$ and
$g: (Y, \rho) \to (Z, m)$
are mappings between metric spaces.
(a) If f is continuous at x & g is continuous at $f(x)$, then $g \circ f: (X, d) \to (Z, m)$ is continuous at x.
(b) If f is cts in X and g is cts. in Y, then $g \circ f$ is cts in X.

(Pf: Easy)

Eg: Let (X, d) be a metric space, $A \subset X$, $A \neq \emptyset$.

Define $\rho_A: X \to \mathbb{R}$ by

$$\rho_A(x) = \inf_{y \in A} d(y, x)$$

(distance from x to the subset A)

Claim: $|\rho_A(x) - \rho_A(y)| \leq d(x, y)$, $\forall x, y \in X$.

Pf of claim: For fixed $x, y \in X$.

By defn. of $\rho_A(y)$

$\forall \varepsilon > 0$, $\exists z \in A$ s.t. $\rho_A(y) + \varepsilon > d(z, y)$

Hence, $\rho_A(x) \leq d(z, x) \leq d(x, y) + d(y, z)$
$< d(x, y) + \rho_A(y) + \varepsilon$

$\Rightarrow$ $\rho_A(x) - \rho_A(y) < d(x, y) + \varepsilon$

Interchanging the roles of x & y

$$\rho_A(y) - \rho_A(x) < d(x, y) + \varepsilon$$

Therefore $|\rho_A(x) - \rho_A(y)| < d(x, y) + \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $|\rho_A(x) - \rho_A(y)| \leq d(x, y)$. #

By claim, $d(x_n, x) \to 0 \Rightarrow \rho_A(x_n) \to \rho_A(x)$

$\therefore \quad \rho_A : (X, d) \to \mathbb{R}$ is cts

(In fact, ρ_A is "Lipschitz continuous")

This example shows that there are "many" cts functions on a metric space.

Notation: Usually, we use the following notations

$$d(x, F) = \inf\{d(x,y) : y \in F\}$$
$$d(E, F) = \inf\{d(x,y) : x \in E, y \in F\}$$

for subsets E & F.

§2.3 Open and Closed Sets

Def: Let (X, d) = metric space

- A set $G \subset X$ is called an open set if $\forall x \in G$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) = \{y : d(y,x) < \varepsilon\} \subset G$.

  (The number $\varepsilon > 0$ may vary depending on x)
- We also define the empty set $\emptyset$ to be an open set.

Prop 2.4: Let (X, d) be a metric space. We have

(a) X and $\emptyset$ are open sets.

(b) Arbitrary union of open sets is open: if G_α, $\alpha \in \mathcal{A}$, is a collection of open sets, then $\bigcup_{\alpha \in \mathcal{A}} G_\alpha$ is an open set.

(c) Finite intersection of open sets is open: if $G_1, \cdots, G_N$ are open sets, then $\bigcap_{j=1}^{N} G_j$ is an open set.

Pf: (a) Clear

(b) Let $x \in \bigcup_{\alpha \in \mathcal{A}} G_\alpha$

$\Rightarrow x \in G_\alpha$ for some $\alpha \in \mathcal{A}$

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset G_\alpha$ (Since G_α open)

$\Rightarrow B_\varepsilon(x) \subset \bigcup_{\alpha \in \mathcal{A}} G_\alpha$

(c) Let $x \in \bigcap_{j=1}^{N} G_j \Rightarrow x \in G_j,\ \forall j = 1, \cdots, N$

$\Rightarrow \exists\, \varepsilon_j > 0$ s.t. $B_{\varepsilon_j}(x) \subset G_j,\ \forall j = 1, \cdots, N.$

Let $\varepsilon = \min\{\varepsilon_1, \cdots, \varepsilon_N\} > 0$. Then

$B_\varepsilon(x) \subset B_{\varepsilon_j}(x) \subset G_j,\ \forall j = 1, \cdots, N$

$\Rightarrow B_\varepsilon(x) \subset \bigcap_{j=1}^{N} G_j$. ※

Def: Let (X, d) be a metric space.
A set $F \subset X$ is called a <u>closed set</u> if the complement $X \setminus F$ is an open set.

Prop 2.5: Let (X, d) be a metric space. We have

(a) X and $\emptyset$ are closed sets.

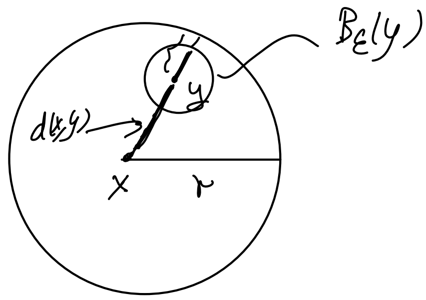
(b) Arbitrary intersection of closed sets is closed: if F_α, $\alpha \in \mathcal{A}$, is a collection of closed sets, then $\bigcap_{\alpha \in \mathcal{A}} F_\alpha$ is a closed set.

(c) Finite union of closed sets is closed: if $F_1, \cdots, F_N$ are closed sets, then $\bigcup_{j=1}^{N} F_j$ is a closed set.

Note Prop 2.4 & 2.5 $\Rightarrow$ X & $\emptyset$ are both open & closed.

eg 2.10 (1) Every metric ball $B_r(x) = \{y \in X : d(x, y) < r\}$ $(r > 0)$ is an open set.

Pf: $\forall\, y \in B_r(x)$



Then $\varepsilon = r - d(x,y) > 0$

& $\forall\, z \in B_\varepsilon(y)$

$$d(z,x) \leq d(z,y) + d(y,x)$$
$$< \varepsilon + d(y,x) = r$$

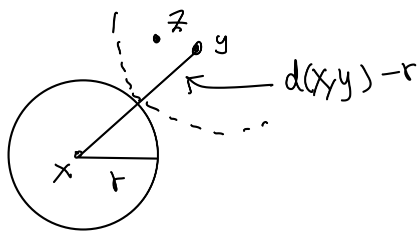
$\Rightarrow \quad B_\varepsilon(y) \subset B_r(x)$ ※

(2) The set $E = \{y \in X : d(y,x) > r\}$ (for a fixed $x \in X$)

is open and hence

$$X \setminus E = \{y \in X : d(y,x) \leq r\} \text{ is closed.}$$

Pf: $\forall\, y \in E$



Then $\varepsilon = d(x,y) - r > 0$

$\forall\, z \in B_\varepsilon(y)$

$$d(z,x) \geq d(x,y) - d(z,y)$$
$$> d(x,y) - (d(x,y) - r) = r$$

$\therefore \quad B_\varepsilon(y) \subset E$ ※

Note: We usually write

$$\overline{B_r}(x) = \overline{B_r(x)} = \{y \in X : d(y,x) \leq r\}$$

the <u>closed ball</u> of radius r centered at x.

(Confusing notation here, may not equal to the "closure" of $B_r(x)$ in a general metric space.)

(3) Since $B_r(x)$ & $E = \{y \in X : d(x,y) > r\}$ are open,

$B_r(x) \cup E$ is open

$\Rightarrow \quad X \setminus (B_r(x) \cup E) = \{y \in X : d(x,y) = r\}$ is closed.

In particular, $E = \{ y \in X : d(x,y) > 0 \}$ is open

$\Rightarrow \{x\} = X \setminus E$ is <u>closed</u> (in any metric space).

(Note: $\{x\}$ may not be open (unless $\exists\, \varepsilon_0 > 0$ s.t. $B_{\varepsilon_0}(x) = \{x\}$))

<u>eg2.11</u> $B_{\frac{1}{n}}(x)$, $n = 1, 2, \cdots$ are open sets

<u>Claim</u> $\bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(x) = \{x\}$ (closed, may not be open)

(even countable infinite intersection of open sets may not be open)

Pf of claim: $\forall\, y \in \bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(x) \Rightarrow y \in B_{\frac{1}{n}}(x)$, $\forall\, n = 1, 2, \cdots$

$\Rightarrow d(y,x) < \frac{1}{n}$, $\forall n$

$\Rightarrow d(y,x) = 0$

$\Rightarrow y = x$ ※