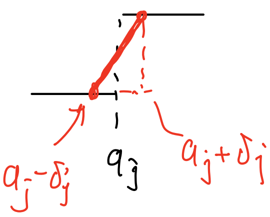
Pf of Thm 1.16 ($\swarrow$ satisfying Lip condition)

Step 1: $\forall \varepsilon > 0$, $\exists$ a 2π-periodic Lip cts function g s.t.

$$\|f - g\|_2 < \varepsilon/2$$

(Ex: Hint: find a step function approximating f as before, and then modify



)

Step 2 Completion of the proof.

Applying Thm 1.7 to the function g in Step 1:

$\exists N > 0$ s.t.

$$\|g - S_N g\|_\infty < \frac{\varepsilon}{2\sqrt{2\pi}}$$

Thus

$$\|g - S_N g\|_2 = \left[\int_{-\pi}^{\pi} (g - S_N g)^2\right]^{1/2} \leq \left[2\pi \|g - S_N g\|_\infty^2\right]^{1/2} = \frac{\varepsilon}{2}$$

By Cor 1.15, $\quad S_N g \in E_N$ too

$$\|f - S_N f\|_2 \leq \|f - S_N g\|_2$$
$$\leq \|f - g\|_2 + \|g - S_N g\|_2 \quad (\text{Ex!})$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (\text{Step 1}).$$

Finally, since $E_N \subset E_n$, $\forall n \geq N$
(∵ more generators) $\quad \in E_N \subset E_n$

we have $\forall n \geq N$, $\|f - S_n f\|_2 \leq \|f - S_N f\|_2 < \varepsilon$

i.e. $\lim_{n\to\infty} \|S_n f - f\|_2 = 0$ ※

Cor 1.17 (a) Suppose that f_1 & f_2 are 2π-periodic integrable functions on $[-\pi,\pi]$ with the same Fourier Series. Then $f_1 = f_2$ almost everywhere.
(i.e. $f_1 = f_2$ except a set of measure zero.)

(b) Suppose that f_1 & f_2 are 2π-periodic continuous functions with the same Fourier series. Then $f_1 = f_2$

Recall: A set E is said to be of measure zero if $\forall \varepsilon > 0$, $\exists$ countably many intervals $\{I_k\}$ s.t.

$$E \subset \bigcup_k I_k \quad \& \quad \sum_k |I_k| < \varepsilon.$$

Pf: (a) Let $f = f_1 - f_2$, then $a_n(f) = b_n(f) = 0 \quad \forall n \geq 0$

$$\Rightarrow S_n f = 0, \forall n \geq 0$$

Hence $\lim_{n \to \infty} \|S_n f - f\|_2 = 0 \Rightarrow \|f\|_2 = 0$

By theory of Riemann integral, $f = 0$ almost everywhere.

(b) We still have $\|f\|_2 = 0$. As f_1, f_2 cts $\Rightarrow f^2$ cts, ≥ 0

$$\Rightarrow f^2 \equiv 0. \quad \#$$

Cor 1.18 (Parseval's Identity)

For every 2π-periodic function f integrable on $[-\pi,\pi]$

$$\|f\|_2^2 = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

where a_0, a_n, b_n are Fourier coefficients of f.

Pf: By def of a_n, b_n

$$\begin{cases} \sqrt{2\pi}\, a_0 = \langle f, \frac{1}{\sqrt{2\pi}} \rangle_2 & \\ \sqrt{\pi}\, a_n = \langle f, \frac{1}{\sqrt{\pi}} \cos nx \rangle_2 & n \geq 1 \\ \sqrt{\pi}\, b_n = \langle f, \frac{1}{\sqrt{\pi}} \sin nx \rangle_2 & \end{cases}$$

orthogonal to the subspace E_N

Then $\langle f, S_N f \rangle_2 = \langle (f - S_N f) + S_N f, S_N f \rangle_2$

(by Cor 1.15)
$$= \langle S_N f, S_N f \rangle_2$$
$$\left(= \|S_N f\|_2^2 \right)$$
$$= \int_{-\pi}^{\pi} \left(a_0 + \sum_{k=1}^{N} a_k \cos kx + b_k \sin kx \right)^2 dx$$
$$= 2\pi a_0^2 + \pi \sum_{k=1}^{N} (a_k^2 + b_k^2)$$

Hence
$$0 \overset{\text{Thm 1.16}}{=\!=} \lim_{N\to\infty} \|f - S_N f\|_2^2$$
$$= \lim_{N\to\infty} \left(\|f\|_2^2 - 2\langle f, S_N f \rangle_2 + \|S_N f\|_2^2 \right)$$
$$= \lim_{N\to\infty} \left(\|f\|_2^2 - \|S_N f\|_2^2 \right)$$

$$\therefore \|f\|_2^2 = \lim_{N\to\infty} \|S_N f\|_2^2 = \lim_{N\to\infty} \left[2\pi a_0^2 + \pi \sum_{k=1}^{N} (a_k^2 + b_k^2) \right]$$

#

eg: By Fourier series of $f_1(x) = x$ on $[-\pi, \pi]$ and Parseval's Identity

(Ex!) $\Rightarrow \quad \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (Euler Formula)

Ch2 Metric Space

In this chapter, X always denotes a non-empty set.

Def: A metric on X is a function
$$d: X \times X \to [0, +\infty) \text{ such that}$$
$\forall\, x, y, z \in X$

(M1) $d(x,y) \geq 0$ & "equality holds $\Leftrightarrow x = y$".

(M2) $d(x,y) = d(y,x)$

(M3) $d(x,y) \leq d(x,z) + d(z,y)$

The pair (X, d) is called a metric space.

Note: Condition (M3) is called the triangle inequality.