§1.4 Weierstrass Approximation Theorem (Application of Thm 1.7)

Recall: A cts function g defined on $[a,b]$ is piecewise linear if $\exists$ a partition $a = a_0 < a_1 < \cdots < a_n = b$ such that g is linear on each subinterval $[a_j, a_{j+1}]$.

Prop 1.11 Let f be a cts function on $[a,b]$. Then $\forall \varepsilon > 0$, $\exists$ a cts, piecewise linear g with

$$g(a) = f(a), \quad g(b) = f(b) \quad \text{such that}$$

$$\|f - g\|_\infty < \varepsilon$$

$$\left(\|f-g\|_\infty = \sup_{[a,b]} |f(x) - g(x)| . \right)$$

Pf: f cts on closed interval $[a,b]$

$\Rightarrow$ f uniform cts on $[a,b]$

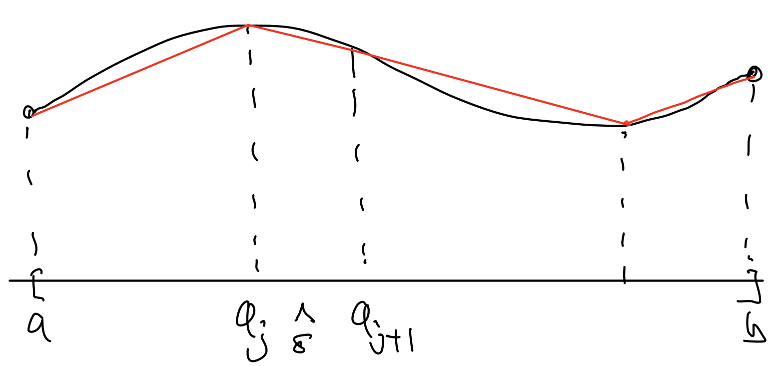
$\Rightarrow$ $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon/2, \quad \forall |x-y| < \delta \qquad (x, y \in [a,b])$$

Partition $[a,b]$ into subintervals $I_j = [a_j, a_{j+1}]$ s.t. $|I_j| = a_{j+1} - a_j < \delta, \ \forall j$.

Define

$$g(x) = \frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j}(x - a_j) + f(a_j), \quad \forall x \in I_j .$$



Clearly $g(a_j) = f(a_j)$, $\forall j$.

In particular $g(a) = f(a)$ & $g(b) = f(b)$.

And $g(x)$ is piecewise linear on $[a,b]$ (and cts)

Then $\forall\, x \in I_j \subset [a,b]$

$$|f(x) - g(x)| = \left| f(x) - \frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j}(x - a_j) - f(a_j) \right|$$

$$\leq |f(x) - f(a_j)| + |f(a_{j+1}) - f(a_j)| \cdot \frac{x - a_j}{a_{j+1} - a_j}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore \quad \sup\limits_{\cup I_j} |f(x) - g(x)| < \varepsilon$, i.e. $\|f - g\|_\infty < \varepsilon$. #

<u>Terminology</u>: A <u>trigonometric polynomial</u> is of the form $P(\cos x, \sin x)$, where $P(x,y)$ is a polynomial of 2-variables

<u>Note</u>: A trigonometric polynomial is a <u>finite</u> Fourier series and vice-versa (Ex!)

<u>Prop 1.12</u> Let f be a cts function on $[0,\pi]$. Then $\forall \varepsilon > 0$ $\exists$ a trigonometric polynomial h s.t. $\|f - h\|_\infty < \varepsilon$.

<u>Pf</u>: Extend f to $[-\pi,\pi]$ by $f(x) = \begin{cases} f(x), & x \in [0,\pi] \\ f(-x), & x \in [-\pi, 0] \end{cases}$ (even extension)

Then this extension is cts on $[-\pi,\pi]$ & $f(\pi) = f(-\pi)$, hence extends to a 2π-periodic <u>cts.</u> function on $\mathbb{R}$

By Prop 1.11, $\forall \varepsilon > 0$, $\exists$ piecewise linear (cts) g on $[-\pi, \pi]$ s.t. $\|f-g\|_\infty < \varepsilon/2$ (sup taking over $[-\pi,\pi]$) and

$$g(\pi) = f(\pi) = f(-\pi) = g(-\pi).$$

$\Rightarrow$ g extends to a piecewise linear 2π-periodic function $\tilde{g}$ on $\mathbb{R}$.

Clearly $\tilde{g}$ satisfies a <u>Lip condition</u> (check!)

Then <u>Thm 1.7</u> $\Rightarrow$ $\exists N > 0$ s.t

$$\|g - S_N g\|_\infty < \varepsilon/2 \qquad (S_N g \to g \text{ uniformly})$$

Therefore,

$$\|f - S_N g\|_\infty \le \|f-g\|_\infty + \|g - S_N g\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore$ $h = S_N g$ is the required trigonometric polynomial. #

<u>Thm 1.13</u> (<u>Weierstrass Approximation Theorem</u>)

Let $f \in C[a,b]$. Then $\forall \varepsilon > 0$, $\exists$ a polynomial q s.t.

$$\|f - q\|_\infty < \varepsilon.$$

Pf: Consider $[a,b] = [0,\pi]$ first.

Extend f to $[-\pi,\pi]$ as in Prop 1.12.

$\forall \varepsilon > 0$, choose trigonometric polynomial $h = P(\cos x, \sin x)$ s.t.

$$\|f - h\|_\infty < \varepsilon/2$$

Using the fact that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$$

converge uniformly.

$\exists N > 0$ s.t.

$$\left\| f(x) - P\left(\sum_{n=0}^{N} \frac{(-1)^n x^{2n}}{(2n)!}, \sum_{n=1}^{N} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \right) \right\|_\infty < \frac{\varepsilon}{2}$$

Clearly $g(x) = P\left(\sum_{n=0}^{N} \frac{(-1)^n x^{2n}}{(2n)!}, \sum_{n=1}^{N} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \right)$

is the required polynomial s.t. $\|f-g\|_\infty < \varepsilon$.

For general $[a,b]$, $\varphi(x) = f\left(\frac{b-a}{\pi} x + a\right) \in C[0,\pi]$

$\Rightarrow \exists\, g(x)$ polynomial s.t.

$$\|\varphi(x) - g(x)\|_\infty < \varepsilon \text{ on } [0,\pi].$$

$\Rightarrow g\left(\frac{\pi}{b-a}(x-a)\right)$ is the polynomial s.t.

$$\left\| f(x) - g\left(\frac{\pi}{b-a}(x-a)\right) \right\|_\infty < \varepsilon \quad . \quad \#$$

§1.5 Mean Convergence of Fourier Series

Notation:

$R[-\pi,\pi]$ = set of Riemann integrable (real) functions on $[-\pi,\pi]$.

Def: (1) $\forall f, g \in R[-\pi,\pi]$, the L²-product (L² inner product) is given by

$$\langle f, g\rangle_2 = \int_{-\pi}^{\pi} f(x)g(x)\,dx$$

(Note: for cpx function $\langle f, g\rangle_2 = \int_{-\pi}^{\pi} f\bar{g}$)

(2) The L²-norm of $f \in R[-\pi,\pi]$ is

$$\|f\|_2 = \sqrt{\langle f, f\rangle_2}$$

(3) The L²-distance between $f, g \in R[-\pi,\pi]$ is

$$\|f-g\|_2 .$$

(4) We said that $f_n \to f$ in L² sense if

$$\|f_n - f\|_2 \to 0 \quad \text{as } n\to\infty .$$

(i.e. $\lim_{n\to\infty} \int_{-\pi}^{\pi} (f_n-f)^2\,dx = 0$, "mean convergence")

Caution: L²-norm & L²-distance on $R[-\pi,\pi]$ are not really "norm" & "distance" in the strict sense as

$$\begin{cases} \|f\|_2 = 0 \not\Rightarrow f=0 \text{ in } R[-\pi,\pi] \\ \|f-g\|_2 = 0 \not\Rightarrow f=g \text{ in } R[-\pi,\pi] \end{cases}$$

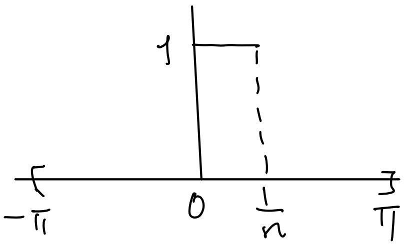
(We only have $\begin{cases} f=0 \text{ almost everywhere} \\ f=g \text{ almost everywhere} \end{cases}$ resp.)

Note: It is not hard to show that

$$f_n \to f \text{ uniformly } \Rightarrow \|f_n - f\|_2 \to 0$$
$$(\|f_n - f\|_\infty \to 0)$$

However $\|f_n - f\|_2 \to 0 \not\Rightarrow f_n \to f$ uniformly!

eg:



$$f_n(x) = \begin{cases} 1, & x \in [0, \frac{1}{n}] \\ 0, & \text{otherwise} \end{cases}$$

Then $\|f_n\|_2^2 = \int_{-\pi}^{\pi} f_n^2 = \frac{1}{n} \to 0$ $\therefore f_n \to 0$ in L^2-sence

But $f_n \not\to 0$ uniformly.

In fact $f_n(x) \longrightarrow \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$

(not even pointwise converge to 0, & the pointwise limit is discts.)

Application to Fourier Series

Consider the functions on $[-\pi, \pi]$

$$\begin{cases} \varphi_0 = \frac{1}{\sqrt{2\pi}} & \text{(const. function)} \\ \varphi_n = \frac{1}{\sqrt{\pi}} \cos nx & (n \geq 1) \\ \psi_n = \frac{1}{\sqrt{\pi}} \sin nx \end{cases}$$

Then

$$\begin{cases} \langle \varphi_m, \varphi_n \rangle_2 = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \\ \langle \varphi_m, \psi_n \rangle_2 = 0, \ \forall m, n \\ \langle \psi_m, \psi_n \rangle_2 = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \end{cases}$$

(check!)

$\therefore \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin nx \right\}_{n=1}^{\infty}$ can be regarded as an "orthonormal basis" in $R[-\pi,\pi]$.

<u>Notation</u> : We denote

$$E_N \overset{\text{def}}{=\!=} \operatorname{span}\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin nx \right\}_{n=1}^{N}$$

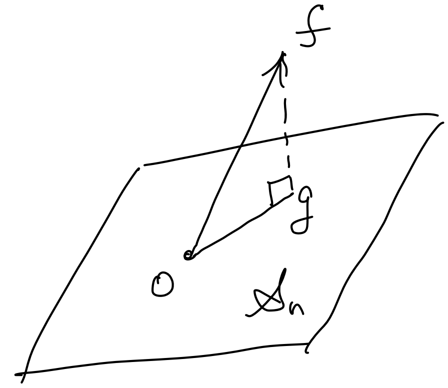
$= (2N+1)$ dim'l vector subspace of $R[-\pi,\pi]$ spanned by the 1^{st} $(2N+1)$ trigonometric functions.

$(\dim E_N = 2N+1)$

In general, if we have an <u>orthonormal set</u> (or <u>orthonormal family</u>)

$\{\phi_n\}_{n=1}^{\infty}$ in $R[-\pi,\pi]$

$\langle \phi_n, \phi_m \rangle_2 = \delta_{mn}$,



we set $S_n = \operatorname{span}\langle \phi_1, \cdots, \phi_n \rangle$

$= n$-dim'l subspace spanned by the 1^{st} n functions in the orthonormal set

Then $\forall f \in R[-\pi,\pi]$, we consider the <u>minimization problem</u>

$$\inf\{ \|f-g\|_2 : g \in S_n \}$$

<u>Prop 1.14</u> : The unique minimizer of $\inf_{g\in S_n} \|f-g\|_2$ is attained at the function

$$g = \sum_{k=1}^{n} \langle f, \phi_k \rangle_2 \, \phi_k \in S_n$$

Pf: Note that minimize $\|f-g\|_2 \Leftrightarrow$ minimize $\|f-g\|_2^2$

Then $\forall g \in \mathcal{S}_n$, $g = \sum_{k=1}^{n} \beta_k \phi_k$ and

$$\|f-g\|_2^2 = \int_{-\pi}^{\pi} |f - \sum_{k=1}^{n} \beta_k \phi_k|^2 \ \underline{\underline{\text{regarded}}}\ \Phi(\beta_1, \cdots, \beta_n) = \Phi(\beta)$$

We first need to show that $\Phi(\beta_1, \cdots, \beta_n) \to \infty$ as $\|\beta\| = \sqrt{\sum \beta_k^2} \to +\infty$

$$\Phi(\beta) = \int_{-\pi}^{\pi} (f - \sum_{k=1}^{n} \beta_k \phi_k)^2$$

$$= \left(\int_{-\pi}^{\pi} f^2\right) - 2 \sum_{k=1}^{n} \beta_k \left(\int_{-\pi}^{\pi} f \phi_k\right) + \sum_{l,k=1}^{n} \beta_k \beta_l \int_{-\pi}^{\pi} \phi_k \phi_l$$

$$= \|f\|_2^2 - 2 \sum_{k=1}^{\infty} \left(\frac{\beta_k}{\sqrt{2}}\right)\left(\sqrt{2}\langle f, \phi_k \rangle_2\right) + \sum_{k=1}^{n} \beta_k^2$$

$$\geq \|f\|_2^2 - \sum_{k=1}^{\infty} \left(\frac{\beta_k^2}{2} + 2\langle f, \phi_k \rangle_2^2\right) + \sum_{k=1}^{n} \beta_k^2$$

$$= \|f\|_2^2 - 2\sum_{k=1}^{\infty} \langle f, \phi_k \rangle_2^2 + \frac{1}{2}\sum_{k=1}^{n} \beta_k^2 \to +\infty$$

as $\|\beta\| \to +\infty$.

$\therefore$ $\Phi(\beta)$ attains a minimum at some $\underline{\text{finite}}$ point $\beta = (\beta_1, \cdots, \beta_n)$

Then easy calculus

$\Rightarrow$ the unique maximum is given by

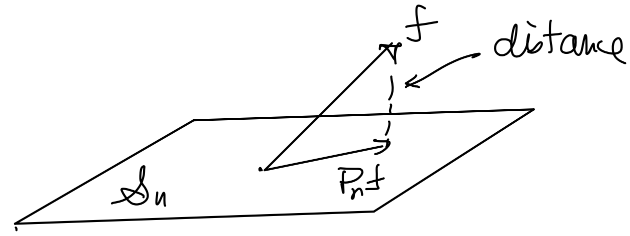
$$\beta_k = \langle f, \phi_k \rangle_2, \ \forall k = 1, \cdots, n.$$

#

<u>Notes</u>: (1) The minimizer $g = \sum_{k=1}^{n} \langle f, \phi_k \rangle_2 \phi_k$ of $\|f-g\|_2$ over $\mathcal{S}_n$ is called the <u>athogonal projection</u> of f onto $\mathcal{S}_n$ & denoted by $P_n f$ ($\in \mathcal{S}_n f$).

(2) $\quad \mathrm{dist}(f, S_n) \; \left(= \inf\{\mathrm{dist}(f,g) : g \in S_n\}\right)$

$= \|f - P_n f\|_2$



Cor 1.15 For 2π-periodic function f integrable on $[-\pi, \pi]$ and $n \geq 1$,

$$\|f - S_N f\|_2 \leq \|f - g\|_2 \quad \forall g \text{ of the form}$$

$$g = \alpha_0 + \sum_{k=1}^{N} (\alpha_k \cos kx + \beta_k \sin kx)$$

with $\alpha_0, \alpha_k, \beta_k \in \mathbb{R}$.

(nth partial sum of the Fourier series of f)

Pf: By def. of Fourier coefficients $S_n f = P_n f$ of the span$\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos kx, \frac{1}{\sqrt{\pi}}\sin kx\right\}_{k=1}^{n}$:

$$\begin{cases} a_0 = \langle f, \frac{1}{\sqrt{2\pi}} \rangle_2 \cdot \frac{1}{\sqrt{2\pi}} \\ a_n \cos nx = \langle f, \frac{1}{\sqrt{\pi}} \cos nx \rangle_2 \cdot \frac{1}{\sqrt{\pi}} \cos nx \\ b_n \sin nx = \langle f, \frac{1}{\sqrt{\pi}} \sin nx \rangle_2 \cdot \frac{1}{\sqrt{\pi}} \sin nx \end{cases} \qquad \text{(Ex!)}$$

#

Thm 1.16 For 2π-periodic (real) function f (Riemann) integrable on $[-\pi, \pi]$, $\quad \lim_{n \to \infty} \|S_n f - f\|_2 = 0$

i.e. the nth partial sum of the Fourier Series of f converges to f in L^2-sense.