Motivation of the definition of Fourier Series

"If" $\quad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \forall x \in \mathbb{R}$

(& assume uniformly convergent.)

Then $\quad \int_{-\pi}^{\pi} f(x) \cos mx\, dx$

$$= a_0 \int_{-\pi}^{\pi} \cos mx\, dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \cos mx\, dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx\, dx \right)$$

It is easy to calculate

$$\left\{ \begin{array}{l} \bullet \ \int_{-\pi}^{\pi} \cos mx\, dx = \begin{cases} 2\pi & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases} \\ \bullet \ \int_{-\pi}^{\pi} \cos nx \cos mx\, dx = \begin{cases} \pi, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases} \\ \bullet \ \int_{-\pi}^{\pi} \sin nx \cos mx\, dx = 0, \quad \forall n, m \geq 1 \end{array} \right.$$

Hence if $m = 0$,

$$\left. \begin{array}{l} LHS = \int_{-\pi}^{\pi} f(x)\, dx \\ RHS = 2\pi a_0 \end{array} \right\} \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

If $m \neq 0$,

$$\left. \begin{array}{l} LHS = \int_{-\pi}^{\pi} f(x) \cos mx\, dx \\ RHS = a_m \pi \end{array} \right\} \quad a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx\, dx$$

Similarly, consider

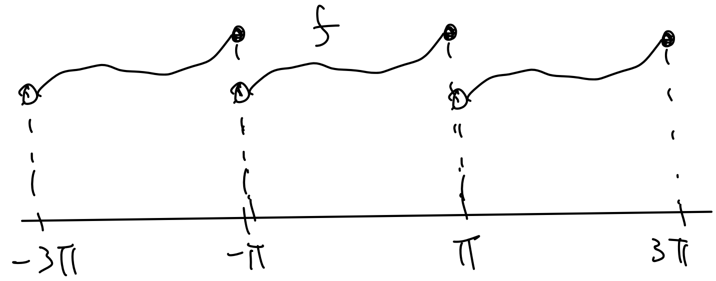
$$\int_{-\pi}^{\pi} f(x) \sin mx\, dx = a_0 \int_{-\pi}^{\pi} \sin mx\, dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \sin mx\, dx + b_n \int_{-\pi}^{\pi} \sin nx \sin mx\, dx \right.$$

and using $\left\{\begin{array}{l} \bullet \ \int_{-\pi}^{\pi} \sin mx\,dx = 0 \quad \forall m \\ \bullet \ \int_{-\pi}^{\pi} \sin nx \sin mx\,dx = \begin{cases} \pi & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases} \end{array}\right.$

$$\Rightarrow \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx, \qquad \forall m\geq 1.$$

<u>Note</u>: For any Riemann integrable function f on $[-\pi,\pi]$, we can define all the a_0, a_n, b_n $(n\geq 1)$ as in the def$\underline{n}$, and hence the <u>Fourier Series</u>.

On the other hand, we can restrict a f to $(-\pi,\pi]$ and extend periodically to a 2π-periodic function $\underline{\underline{\tilde{f}}}$ on $\mathbb{R}$



And according to the def$\underline{n}$. of Fourier coefficients, f & $\tilde{f}$ have the <u>same</u> Fourier Series!

So we will <u>not</u> <u>distinguish</u> f & $\tilde{f}$.

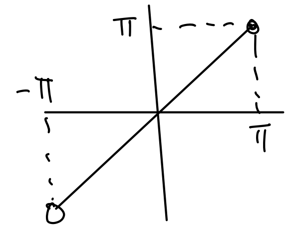
<u>Notation</u> We use $\quad f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

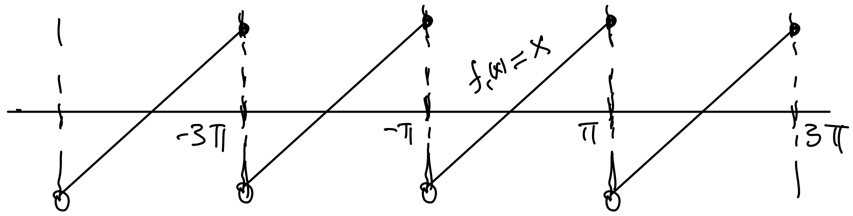
means "the trigonometric series on the RHS is the Fourier Series of f".

(does not indicate the series converges to f in any sense.)

eg 1.1 $f_1(x) = x$ restricted to $(-\pi, \pi]$

Extension to 2π-periodic function $\tilde{f}_1$ on $\mathbb{R}$





$$\begin{cases} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0 \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = 0 \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = (-1)^{n+1}\frac{2}{n} \end{cases}$$ (check!)

$$\therefore\ f_1(x) = x \sim \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n}\sin nx$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx$$ (is a sine series)

($\because f_1$ is odd)

Notes: (1) For $x = \pm\pi$, Fourier series$\Big|_{\pm\pi} = 0$

But $\left.\begin{matrix} f_1(\pm\pi) = \pm\pi \\ \tilde{f}_1(\pm\pi) = \pi \end{matrix}\right\} \neq$ Fourier Series$\Big|_{\pm\pi} = 0$

(2) Convergence is not clear (for $x \neq \pm\pi$)

as the terms decay like $\frac{1}{n}$ & $\sum \frac{1}{n}$ doesn't converge.

Notation "Big O" & "small o"

Let $\{x_n\}$ be a sequence, then

(i) $x_n = O(n^s) \Leftrightarrow |x_n| \le Cn^s$ for some const. $C > 0$

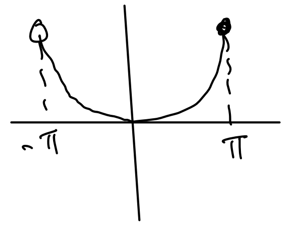
(ii) $x_n = o(n^s) \Leftrightarrow |x_n|/n^s \to 0$ as $n \to \infty$.

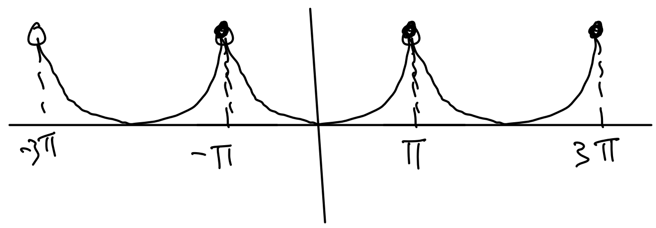
egs: (i) $x_n = \frac{2(-1)^{n+1}}{n}\sin nx = O(n^{-1}) = O(\frac{1}{n})$, $\left(|x_n| \leq \frac{2}{n}\right)$

(ii) $x_n = \log n = o(n)$ $\left(\frac{\log n}{n} \to 0 \text{ as } n \to \infty \right)$

<u>Eg 1.2</u> $f_2(x) = x^2$ restricted to $(-\pi, \pi]$

Extension to a 2π-periodic function $\tilde{f}_2$ on $\mathbb{R}$





$\tilde{f}_2$ is continuous (since $f_2(-\pi) = f_2(\pi)$)

$\tilde{f}_2$ is an even function

It is an <u>easy exercise</u> of integration to find that

$$f_2(x) = x^2 \sim \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\cos nx \qquad (\text{Ex!})$$

(cosine series, f_2 even)

One sees that $a_n = O(\frac{1}{n^2}) \Rightarrow \sum |a_n| < \infty$

$\Rightarrow$ Fourier series converges uniformly to a continuous function.

(Will it be the function $\tilde{f}_2$? See later discussion)

<u>Observation</u>: Egs 1 & 2: $\begin{cases} \text{odd function} \to \text{sine series} \\ \text{even function} \to \text{cosine series} \end{cases}$

This is true in general! (Ex!)

Complex Fourier Series

Def: (1) A complex trigonometric series is a series of the form

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

($\{c_n\}_{-\infty}^{\infty}$ is called a bisequence of cpx numbers & $\{c_n e^{inx}\}_{n=-\infty}^{\infty}$ is a bisequence of cpx-valued functions)

(2) $\sum_{-\infty}^{\infty} c_n e^{inx}$ is said to be convergent at x if

$$\lim_{N\to+\infty} \sum_{n=-N}^{N} c_n e^{inx} \text{ exists.}$$

Def: Complex Fourier Series of a 2π-periodic cpx-valued function f which is integrable on $[-\pi,\pi]$, denoted by

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

is a cpx trigonometric series with (cpx) Fourier coefficients c_n defined by

$$\boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx, \quad \forall n \in \mathbb{Z}}$$

Motivation for cpx Fourier Series:

"If" $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$ & "converges nicely"

Then $f(x)e^{-imx} = \sum_{-\infty}^{\infty} c_n e^{i(n-m)x}$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) e^{-imx}\,dx = \sum_{-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx$$

It is easy to find
$$\int_{-\pi}^{\pi} e^{i(n-m)x}\,dx = \begin{cases} 2\pi, & \text{if } n=m \\ 0, & \text{if } n\neq m \end{cases}$$

$$\Rightarrow \quad \int_{-\pi}^{\pi} f(x) e^{-imx}\,dx = c_m \cdot 2\pi \quad \#$$

<u>Relationship</u> between (Real) Fourier Series & Cpx Fourier Series for a <u>real-valued</u> function <u>f</u>.

By
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i\sin nx)\,dx$$
$$= \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\right) - \frac{1}{2}i\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right)$$

Therefore

for $n=0$, $\quad c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = a_0$

for $n\geq 1$, $\quad c_n = \frac{a_n}{2} - i\frac{b_n}{2}$

for $n\leq -1$, then $(-n)\geq 1$ &

$$c_n = \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos((-n)x)\,dx\right) + \frac{1}{2}i\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(-n)x\,dx\right)$$
$$= \frac{1}{2}a_{(-n)} + i\,\frac{1}{2}b_{(-n)}$$

$\therefore$
$$\boxed{c_n = \begin{cases} \frac{1}{2}(a_n - ib_n), & \text{for } n\geq 1 \\ a_0, & \text{for } n=0 \\ \frac{1}{2}(a_{(-n)} + ib_{(-n)}), & \text{for } n\leq -1 \end{cases}}$$

for real-valued function.

Corollary: If f is a real-valued function, then

$$C_{-n} = \overline{C_n} \quad \text{cpx conjugate}, \ \forall n \in \mathbb{Z}$$

(i.e. $C_n = \overline{C_{-n}}$)

(Pf: Easy)

Prop: Let f be a 2π-periodic real-valued function which is differentiable on $[-\pi, \pi]$ with f' integrable on $[-\pi, \pi]$.

Denote the Fourier coefficients of f & f' by

$\{a_n(f), b_n(f) ; C_n(f)\}$ & $\{a_n(f'), b_n(f') ; C_n(f')\}$ respectively.

Then

$$\boxed{\begin{aligned} a_n(f') &= n\, b_n(f) \\ b_n(f') &= -n\, a_n(f) \\ \& \quad C_n(f') &= in\, C_n(f) \end{aligned}}$$

(So it is more convenient to work with cpx Fourier coefficients when derivatives involved.)

Pf:
$$a_n(f') = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\, dx$$

$$(\text{integration by part}) = \frac{1}{\pi}\left[f(x)\cos nx \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)(-n\sin nx)\, dx \right]$$

$$(f(\pi) = f(-\pi)) = \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = n\, b_n(f)$$

Similarly for $b_n(f') = -n\, a_n(f)$ (Check!)

For $C_n(f')$, either from the above formula relating C_n to a_n & b_n, or integration by part directly

$$C_n(f') = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}\, dx = \frac{1}{2\pi}\left[\cancel{f(x)e^{-inx}\Big|_{-\pi}^{\pi}}^{\,0} + (in)\int_{-\pi}^{\pi} f(x)e^{-inx}\, dx \right] = in\, C_n(f)$$

※

Remarks: (1) f is differentiable on $[-\pi,\pi]$ doesn't implies f' is (Riemann) integrable on $[-\pi,\pi]$. So the conditions in the Prop are needed.

A counterexample can be constructed from the example

$$g(x) = \begin{cases} x^{\frac{3}{2}} \sin\frac{1}{x}, & x>0 \\ 0, & x=0 \end{cases}$$

Then $$g'(x) = \begin{cases} \frac{3}{2}x^{\frac{1}{2}}\sin\frac{1}{x} - \frac{1}{x^{1/2}}\cos\frac{1}{x}, & x>0 \\ 0, & x=0. \end{cases}$$

Note that $g'(x)$ is unbounded, it is not Riemann integrable on any closed interval $[0,\varepsilon]$ $(\varepsilon>0)$.

(2) However, if f is continuously differentiable on $[-\pi,\pi]$, then f' is cts on $[-\pi,\pi]$ & hence (Riemann) integrable on $[-\pi,\pi]$.

Fourier Series of $2T$-periodic (real) functions

Let f be a $2T$-periodic function

Then $g(x) = f(\frac{T}{\pi}x)$ is 2π-periodic

Therefore

$$f(\tfrac{T}{\pi}x) = g(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

with

$$\begin{cases} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)dx = \frac{1}{2T}\int_{-T}^{T} f(y)dy \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx\,dx = \frac{1}{T}\int_{-T}^{T} f(y)\cos(\frac{n\pi}{T}y)dy \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx\,dx = \frac{1}{T}\int_{-T}^{T} f(y)\sin(\frac{n\pi}{T}y)dy \end{cases}$$

$$f(y) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{T} y\right) + b_n \sin\left(\frac{n\pi}{T} y\right) \right]$$

with

$$\begin{cases} a_0 = \dfrac{1}{2T} \displaystyle\int_{-T}^{T} f(y)\,dy \\ a_n = \dfrac{1}{T} \displaystyle\int_{-T}^{T} f(y) \cos\left(\frac{n\pi}{T} y\right) dy \\ b_n = \dfrac{1}{T} \displaystyle\int_{-T}^{T} f(y) \sin\left(\frac{n\pi}{T} y\right) dy \end{cases} \quad n \geq 1$$

is called Fourier series of the 2T-periodic function f.