§6.3 Schwarz Lemma and Conformal Self-Maps

Thm 1 (Schwarz Lemma)

Suppose that $f(z)$ is an analytic function on $\{|z|<1\}$ such that $f(0)=0$ and $|f(z)|\leq 1$, $\forall |z|<1$.

Then $(*)_1$: $|f(z)| \leq |z|$, $\forall |z|<1$, and

$(*)_2$: $|f'(0)| \leq 1$.

Furthermore, if $|f'(0)|=1$ or $|f(z_0)|=|z_0|$ for some z_0 with $0<|z_0|<1$, then

$$f(z) = e^{i\alpha} z \quad \text{for some constant } \alpha\in\mathbb{R}.$$

Pf: Define $g(z)$ in $\{|z|<1\}$ by

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & \text{for } z\neq 0 \\ f'(0) & \text{for } z=0 \end{cases}.$$

Clearly, $g(z)$ analytic in $0<|z|<1$.

And $f'(0)$ exists & $f(0)=0$ $\Rightarrow \left| \lim_{z\to 0} \frac{f(z)}{z} \right| \leq C$ for $|z|\sim 0$

$\Rightarrow |f(z)| \leq C_1 |z|$ for $|z|\sim 0$

$\Rightarrow |g(z)| \leq C_1$ near $z=0$.

$\therefore$ $z=0$ is a removable singular point.

$\therefore$ $g: \{|z|<1\} \to \mathbb{C}$ is analytic.

$\forall z \in \{|z|<1\}$, choose $r_0 < 1$ such that

$$|z| < r_0 < 1.$$

Then by maximum modulus principle

$$|g(z)| \leq \max_{\{|\xi| \leq r_0\}} |g(\xi)| = \max_{\{|\xi| = r_0\}} |g(\xi)|$$

$$= \max_{\{|\xi| = r_0\}} \frac{|f(\xi)|}{|\xi|} \leq \frac{1}{r_0}$$

Letting $r_0 \to 1$, we have $|g(z)| \leq 1$.

Since z is arbitrary, we've proved that

$$\begin{cases} \dfrac{|f(z)|}{|z|} \leq 1 & \text{if } z \neq 0 \\ |f'(0)| \leq 1 & \text{if } z = 0 . \end{cases}$$

Now suppose that $|f'(0)| = 1$ or $|f(z_0)| = |z_0|$ for some z_0 in $\{0 < |z_0| < 1\}$. Then either $|g(0)| = 1$ or $|g(z_0)| = 1$. In both cases, $|g(z)|$ attains interior maximum point. Hence

maximum modulus principle $\Rightarrow$ $g(z) = \text{const}$.
with modulus 1. Therefore $g(z) = e^{i\alpha}$, for some $\alpha \in \mathbb{R}$. $\Rightarrow$ $f(z) = e^{i\alpha} z$, $\forall\ 0 \le |z| < 1$

$\#$

Thm 2 $f : \{|z|<1\} \to \{|z|<1\}$ is a conformal (i.e. analytic & (locally) 1-1) self-map of $\{|z|<1\}$.

$$\Longleftrightarrow \quad \boxed{f(z) = e^{i\theta_0} \frac{z-a}{1-\bar{a}z} \quad \text{for some } a \in \mathbb{C},\ |a|<1, \text{ and } \theta_0 \in \mathbb{R}}$$

Pf: ($\Leftarrow$) Clearly $f(z)$ is a linear fractional transformation

$\therefore$ $f(z)$ is 1-1. As $\left|\frac{1}{\bar{a}}\right| = \frac{1}{|a|} > 1$, the pole is outside $\{|z|<1\}$.

$\therefore$ $f(z)$ is analytic in $\{|z|<1\}$

For $z = e^{i\theta}$, we have

$$|f(e^{i\theta})| = |e^{i\theta_0}| \frac{|e^{i\theta} - a|}{|1 - \bar{a} e^{i\theta}|} = \frac{|e^{i\theta}|\,|1 - ae^{-i\theta}|}{|1 - \overline{(ae^{-i\theta})}|}$$

$$= \frac{|1 - ae^{-i\theta}|}{|\overline{1 - ae^{-i\theta}}|} = 1.$$

$\therefore$ f maps boundary circle $\{|z|=1\}$ to $\{|z|=1\}$.

Note that $|a|<1$ and $f(a)=0$.

$\therefore$ f has an interior point $z=a$ maps to the interior of $\{|z|<1\}$. Therefore $f(z)$ maps $\{|z|<1\}$ onto $\{|z|<1\}$ since f 1-1 & continuous.

(This can also be seen by explicit calculation of the inverse mapping:

$$f^{-1}(z) = e^{-i\theta_0} \cdot \frac{z-(-ae^{i\theta_0})}{1-\overline{(-ae^{i\theta_0})}z}.$$)

$\therefore$ f is a conformal self-map of $\{|z|<1\}$.

($\Rightarrow$) Conversely, let f be a conformal self-map of $\{|z|<1\}$. Let $a=f(0)\in\{|z|<1\}$

Consider $$g(z)=\frac{f(z)-a}{1-\bar{a}f(z)}.$$

Clearly:
- $g(z)$ analytic in $\{|z|<1\}$ $\left(|a|=|f(0)|<1\right)$
- $g(0)=\dfrac{f(0)-a}{1-\bar{a}f(0)}=0$

- $|g(z)| = \left|\dfrac{f(z)-a}{1-\bar{a}f(z)}\right| < 1$, $\forall |z|<1$

(by the proof of the part ($\Leftarrow$)).

By Schwarz Lemma, $|g'(0)| \leq 1$.

Note that by the proof of the part ($\Leftarrow$),

$g(z)$ is conformal self-map of $\{|z|<1\}$.

Hence $g^{-1}: \{|z|<1\} \to \{|z|<1\}$ exists and also a conformal self-map of $\{|z|<1\}$.

In particular, $|g^{-1}(z)| < 1$, $\forall |z|<1$.

And $g(0)=0 \Rightarrow g^{-1}(0)=0$

$\therefore$ Schwarz Lemma again $\Rightarrow |(g^{-1})'(0)| \leq 1$.

Using $(g^{-1})'(0) = \dfrac{1}{g'(0)}$, we have $\dfrac{1}{|g'(0)|} \leq 1$.

Hence $|g'(0)| = 1$. Equality case of Schwarz Lemma holds,

$\therefore$ $g(z) = e^{i\alpha} z$, $\forall |z|<1$.

i.e. $$\frac{f(z)-a}{1-\bar{a}f(z)} = e^{i\alpha} z \quad , \quad \forall \ |z|<1$$

$$\Rightarrow \qquad f(z) = e^{i\alpha} \cdot \frac{z-(-ae^{-i\alpha})}{1-\overline{(-ae^{-i\alpha})}\, z}$$

is of the required form. ⁂

<u>Recall</u> that $w = \varphi(z) = i\,\frac{1-z}{1+z}$ maps $\{|z|<1\}$ conformally onto $\{x+iy : y>0\}$. So for any conformal self-map

$f : \{|z|<1\} \to \{|z|<1\}$,

$$\begin{tikzcd} \{|z|<1\} \arrow[r, "f"] \arrow[d, "\varphi"'] & \{|z|<1\} \arrow[d, "\varphi"] \\ \{x+iy : y>0\} \arrow[r, "g"] & \{x+iy : y>0\} \end{tikzcd}$$

$$g(z) \overset{\text{def}}{=} \varphi \circ f \circ \varphi^{-1}(z)$$

is a conformal self-map of $\{x+iy : y>0\}$ the upper half-plane. Conversely, if g is a conformal self-map of $\{x+iy : y>0\}$, then

$$f(z) \overset{\text{def}}{=} \varphi^{-1} \circ g \circ \varphi$$

is a conformal self-map of $\{|z|<1\}$, hence of the form

$$\varphi^{-1}\circ g\circ\varphi(z)=e^{i\theta_0}\frac{z-a}{1-\bar{a}z}\quad\left(\theta_0\in\mathbb{R},\ \begin{matrix}a\in\mathbb{C}\\|a|<1\end{matrix}\right)$$

$\therefore$ g is a conformal self-map of $\{x+iy : y>0\}$

$\Leftrightarrow$ g is a linear fractional transformation that maps upper half-plane to upper half plane.

In particular, g maps real-axis onto real-axis.

<u>Thm3</u> $g:\{x+iy:y>0\}\to\{x+iy:y>0\}$ is a conformal self-map

$\Leftrightarrow$ $$\boxed{g(z)=\frac{az+b}{cz+d}\ \text{ with } a,b,c,d\in\mathbb{R},\ ad-bc>0}$$

and can be normalized to

$$\boxed{g(z)=\frac{az+b}{cz+d}\ \text{ with } a,b,c,d\in\mathbb{R},\ ad-bc=1}$$

(Pf: Ex! Hint: Use implicit form in terms of cross-ratio.)

Remark: By Thm 4 & 5, we have 3 real-dimensions of freedom to choose our conformal self-maps in both cases:

- $\{|z|<1\}$: $\theta_0 \in \mathbb{R}$, $a \in \{|z|<1\}$
- $\{x+iy : y>0\}$: $a,b,c,d \in \mathbb{R}$ with 1 constraint $ad-bc=1$.

Cor 1: Given 2 sets of distinct 3 points $\{x_1, x_2, x_3\}$ and $\{u_1, u_2, u_3\}$ on the real axis ordered in counterclockwise sense, there exists a unique conformal self-map $g: \{x+iy : y>0\} \to \{x+iy : y>0\}$ such that

$$g(x_i) = u_i \quad , \forall i = 1,2,3.$$

Remarks:
- One of the x_i, or one of the u_i, could be ∞.
- counterclockwise sense: $x_1 < x_2 < x_3$, $x_3 < x_1 < x_2$, or $x_2 < x_3 < x_1$.

Pf: We may assume $\{u_1, u_2, u_3\} = \{0, 1, \infty\}$.

If this case holds, then for general $\{u_i\}$, we can use this special case in the following way:

$$\begin{tikzcd}
(x_1, x_2, x_3) \arrow[r] \arrow[rr, bend right] & (0, 1, \infty) & (u_1, u_2, u_3) \arrow[l]
\end{tikzcd}.$$

For the special case,

$$g(z) = \frac{z - x_1}{z - x_3} \cdot \frac{x_2 - x_3}{x_2 - x_1}$$

$$= \frac{tz - tx_1}{z - x_3} \quad \text{where } t = \frac{x_2 - x_3}{x_2 - x_1} \text{ is "real".}$$

If $x_3 = \infty$, $g(z) = az + b$ for some a, b, and easy to handle. (Ex!)

If $x_3 \neq \infty$, then $g(z)$ is of the form $a, b, c, d \in \mathbb{R}$

and
$$ad - bc = t(-x_3) - (-tx_1) \cdot 1$$
$$= t(x_1 - x_3)$$
$$= \frac{(x_2 - x_3)(x_1 - x_3)}{x_2 - x_1} > 0$$

since $x_1 < x_2 < x_3$, $x_3 < x_1 < x_2$, or $x_2 < x_3 < x_1$.

$\therefore$ g is the required conformal self-map as clearly

$$g(x_1) = 0, \quad g(x_2) = 1, \quad g(x_3) = \infty.$$

Finally, uniqueness follows from general uniqueness of linear fractional transformations. $\#$

<u>Cor 2</u>: Given 2 sets of distinct 3 points $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ on the boundary unit circle $\{|z|=1\}$ ordered in counterclockwise sense, there exist a unique conformal self-map $f: \{|z|<1\} \to \{|z|<1\}$ such that

$$f(z_i) = w_i, \quad \forall i = 1, 2, 3.$$

(Pf: Transform $\{|z|<1\}$ onto $\{x+iy: y>0\}$ & use Cor 1.)

§6.4 Normal Families

Thm 1 (Weierstrass Theorem)

Let $\{f_n(z)\}_{n=1}^{\infty}$ be a sequence of analytic functions on a domain D. If f_n converges uniformly on every compact subset of D to a function f, then f is analytic on D. Furthermore, the sequence $\{f_n'\}_{n=1}^{\infty}$ converges uniformly to f' on every compact subset.

Pf: Since D is open, for any $z_0 \in D$, we can find

$r = r(z_0) > 0$ such that $\{|z - z_0| \leq r\} \subset D$.

For any closed contour γ in $\{|z - z_0| < r\} \subset \{|z-z_0| \leq r\}$

$$\int_\gamma f_n(z)\,dz = 0, \quad \forall n, \text{ as } f_n \text{ are analytic}$$

in D. By assumption, $f_n \to f$ uniformly on $\{|z-z_0| \leq r\}$.

Therefore, $$\int_\gamma f(z)\,dz = \int_\gamma \lim_{n\to\infty} f_n(z)\,dz \underset{\text{(uniform convergence)}}{=} \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = 0$$

Since γ is an arbitrary closed curve in $\{|z - z_0| < r\}$,

f is analytic in $\{|z-z_0|<r\}$.

(by Thm 2 of §3.9, usually called Morera's Thm)

Particularly, f is analytic at z_0.

Since $z_0 \in D$ is arbitrary, f is analytic in D.

To prove that $f_n' \to f'$ uniformly on compact subset, we only need to show that $\forall\, z_0 \in D$, $\exists\, \delta > 0$ s.t.

$f_n' \to f'$ uniformly on $|z-z_0|<\delta$.

As in the above, one choose $r>0$ s.t. $\{|z-z_0|\le r\} \subset D$

then take $\delta = \frac{r}{2} > 0$.

Note that $\forall\, \xi \in \{|z-z_0|<\delta\}$, we have

$$\{|z-\xi| \le \delta\} \subset \{|z-z_0| \le r\}.$$

Hence f_n analytic on $\{|z-\xi|\le\delta\}$ and Cauchy Integral Formula implies

$$f_n'(\xi) = \frac{1}{2\pi i} \int_{|z-\xi|=\delta} \frac{f_n(z)}{(z-\xi)^2}\, dz$$

and $$f'(\zeta) = \frac{1}{2\pi i} \int_{|z-\zeta|=\delta} \frac{f(z)}{(z-\zeta)^2} dz$$

$$\therefore \left| f_n'(\zeta) - f'(\zeta) \right| \leq \frac{1}{2\pi} \left| \int_{|z-\zeta|=\delta} \frac{f_n(z) - f(z)}{(z-\zeta)^2} dz \right|$$

$$\leq \frac{1}{2\pi} \left(\sup_{|z-\zeta|=\delta} |f_n(z) - f(z)| \right) \frac{1}{\delta^2} \cdot 2\pi\delta$$

$$= \frac{1}{\delta} \sup_{|z-\zeta|=\delta} |f_n(z) - f(z)| .$$

Then $f_n \to f$ uniformly in $\{|z-z_0| \leq r\} \Rightarrow$

$\forall \varepsilon > 0, \exists N > 0$ (indep. of z in $\{|z-z_0| \leq r\}$)

s.t. $\forall n \geq N$,

$$|f_n(z) - f(z)| < \varepsilon, \quad \forall |z - z_0| \leq r.$$

Hence, $\forall \varepsilon > 0, \exists N > 0$ s.t.

$$\forall n \geq N \quad |f_n'(\zeta) - f'(\zeta)| \leq \frac{\varepsilon}{\delta}, \quad \forall \zeta \in \{|z-z_0| < \delta\}.$$

$\therefore f_n' \to f'$ uniformly on $\{|z-z_0| < \delta\}$. #

Def 1 : A family $\mathcal{F}$ of analytic functions on a domain D is said to be normal if every sequence of $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset.

Def 2 : A family $\mathcal{F}$ of analytic functions on a domain D is said to be uniformly bounded on compact subsets of D if $\forall$ compact subset $K \subset D$, there exists $M > 0$ (M may depends on K) such that

$\forall f \in \mathcal{F}$, we have $|f(z)| \le M$, $\forall z \in K$.

Def 2' : A family $\mathcal{F}$ of continuous functions on a domain D is said to be equibounded on a subset $E \subset D$ if there exists $M > 0$ (may depends on E) such that

$\forall f \in \mathcal{F}$, we have $|f(z)| \le M$, $\forall z \in E$.

Def 3 : A family $\mathcal{F}$ of continuous functions on a domain D is said to be equicontinuous on a subset $E \subset D$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ (may depends on E) such that $\forall f \in \mathcal{F}$, we have

$$|z - w| < \delta \implies |f(z) - f(w)| < \varepsilon.$$
$$(z, w \in E)$$

Thm 2 (Arzela-Ascoli) Let K be a compact set and $\mathcal{F}$ be family of continuous functions on K which is equibounded and equicontinuous. Then $\mathcal{F}$ contains a sequence $\{f_n\}$ which converges uniformly on K.

(Pf: Omitted. See standard text book in Analysis)