HW 1 Due Jan 26, 2017

1. Define $e^z = e^x(\cos y + i\sin y)$ for $z = x + iy$.

(a) Proof that e^z is differentiable everywhere by calculating the partial derivatives of the real and imaginary parts and checking the Cauchy-Riemann equations.

(b) Show the following properties of e^z:

(i) $|e^z| = e^x$

(ii) $\arg e^z = y + 2n\pi,\ n \in \mathbb{Z}$

(iii) $e^z \neq 0,\ \forall z \in \mathbb{C}$

(iv) $e^{z_1}e^{z_2} = e^{z_1+z_2}$

(v) $\frac{d}{dz}e^z = e^z$ (Hint: use part (a))

(vi) $e^{z+2\pi i} = e^z$ and $e^{2\pi i} = 1$.

2. Define for all $z \in \mathbb{C}$,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

(a) Using Euler formula, $\cos x$ and $\sin x$, for $z = x \in \mathbb{R}$, are the original defined trigonometric functions for real numbers.

(b) Show the following properties of $\cos z$ and $\sin z$:

(i) $\frac{d}{dz}\sin z = \cos z$, $\frac{d}{dz}\cos z = -\sin z$

(ii) $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$

(iii) $e^{iz} = \cos z + i\sin z$ (generalization of Euler's formula)

(iv) $\begin{cases} \sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \\ \cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \end{cases}$

(v) $\sin^2 z + \cos^2 z = 1$

(vi) For $z = x+iy$,

$$\begin{cases} \sin z = \sin x \cosh y + i \cos x \sinh y \\ \cos z = \cos x \cosh y - i \sin x \sinh y, \end{cases}$$

where $\cosh y = \frac{e^y + e^{-y}}{2}$ and $\sinh y = \frac{e^y - e^{-y}}{2}$.

(vii) For $z = x+iy$

$$\begin{cases} |\sin z|^2 = \sin^2 x + \sinh^2 y \\ |\cos z|^2 = \cos^2 x + \sinh^2 y \end{cases}$$

(c) Do we still have $|\sin z| \le 1$ and $|\cos z| \le 1$ for $z \in \mathbb{C}$? Justify your answer.

(d) Find all complex numbers $z \in \mathbb{C}$ such that $\sin z = 0$. And do the same for $\cos z = 0$.

3. Define for all $z \in \mathbb{C}$,

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^{z} - e^{-z}}{2}$$

(<u>hyperbolic cosine</u> and <u>hyperbolic sine</u> respectively)

Show that

(a) $\frac{d}{dz} \cosh z = \sinh z$, $\frac{d}{dz} \sinh z = \cosh z$

(b) $$\begin{cases} \sinh(iz) = i \sin z, & \cosh(iz) = \cos z \\ \sin(iz) = i \sinh z, & \cos(iz) = \cosh z \end{cases}$$

(c) $\sinh(-z) = -\sinh z$, $\cosh(-z) = \cosh z$

(d) $\cosh^2 z - \sinh^2 z = 1$

(e) $$\begin{cases} \sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2 \\ \cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \end{cases}$$

(f) $$\begin{cases} \sinh z = \sinh x \cos y + i \cosh x \sin y \\ \cosh z = \cosh x \cos y + i \sinh x \sin y \end{cases}$$

(g) $$\begin{cases} |\sinh z|^2 = \sinh^2 x + \sin^2 y \\ |\cosh z|^2 = \sinh^2 x + \cos^2 y \end{cases}$$