## §2.3 Limits

**Def**: The function $f(z)$ has a limit $w_0$ as $z$ approaches $z_0$, denoted by

$$\lim_{z \to z_0} f(z) = w_0$$

means $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|f(z) - w_0| < \varepsilon, \quad \forall\ 0 < |z - z_0| < \delta.$$

**Note**: Using mapping representation $(u,v) = f(x,y)$ and note that $|f(z) - w_0|$ = Euclidean distance between the points $f(z)$ & $w_0$, we see that the above is equivalent to

$$\lim_{(x,y) \to (x_0,y_0)} (u(x,y), v(x,y)) = (u_0, v_0)$$

where $w_0 = u_0 + i v_0$, $z_0 = x_0 + i y_0$.

**Thm**: If $\lim_{z \to z_0} f(z)$ exists, it is unique.

**eg**: $\displaystyle \lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{|z| \to 0} \frac{re^{i\theta}}{re^{-i\theta}} = \lim_{|z| \to 0} e^{i2\theta}$

$\begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix}$ $\displaystyle = \lim_{(x,y) \to (0,0)} (\cos\theta, \sin\theta)$ doesn't exist.

## §2.4 Theorems on Limits

**Thm 1** Suppose $f(z) = u(x,y) + iv(x,y)$, $z = x+iy$

and $z = x_0 + iy_0$, $w_0 = u_0 + iv_0$

Then

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \ \& \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

$$\Longleftrightarrow \quad \lim_{z\to z_0} f(z) = w_0$$

**Thm 2** Suppose that $\lim_{z\to z_0} f(z) = w_0$, $\lim_{z\to z_0} g(z) = \zeta_0$

Then

(1) $\lim_{z\to z_0} \left[f(z) \pm g(z)\right] = w_0 + \zeta_0$

(2) $\lim_{z\to z_0} \left[f(z)\, g(z)\right] = w_0 \zeta_0$

(3) If $\zeta_0 \neq 0$, $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{w_0}{\zeta_0}$

## §2.5 Convergence of Sequences

**Def**: (i) An infinite sequence $\{z_n\}_{n=1}^{\infty}$ of cpx numbers has a limit $z$ if $\forall \varepsilon > 0$, $\exists$ positive integer $n_0$ such that $|z_n - z| < \varepsilon$, $\forall n > n_0$

(ii) When limit $z$ exists, the sequence is said to converge to $z$

and denoted by $\lim_{n\to\infty} z_n = z$.

(iii) If a sequence has no limit, it <u>diverges</u>.

<u>Note</u>: If limit exists, it is unique.

<u>Thm</u> Suppose that $z_n = x_n + i y_n$ $(n = 1, 2, 3, \cdots)$ &

$z = x + iy$.

Then $\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x$ & $\lim_{n\to\infty} y_n = y$.

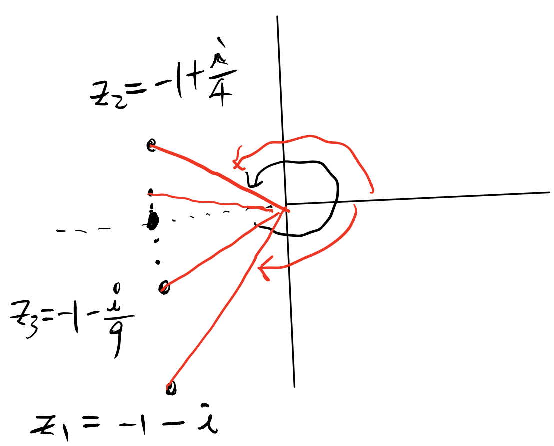
By the thm, we can write

$$\boxed{\lim_{n\to\infty} (x_n + i y_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n}$$

and

$$\boxed{\lim_{n\to\infty} |z_n| = \left| \lim_{n\to\infty} z_n \right|}$$

<u>eg</u>: $\lim_{n\to\infty} \underbrace{\left(-1 + i \frac{(-1)^n}{n^2}\right)}_{z_n} = -1 + i \lim_{n\to\infty} \frac{(-1)^n}{n^2} = -1.$




Principal argument of $z_n$

$= \operatorname{Arg} z_n$

$\therefore$ $\operatorname{Arg} z_{2k+1} \to -\pi$

$\operatorname{Arg} z_{2k} \to \pi$

$\Rightarrow \lim_{n\to\infty} \operatorname{Arg} z_n$ doesn't exist.

Summary: If $z_n \to z$, then

$$\begin{cases} \operatorname{Re} z_n \to \operatorname{Re} z \\ \operatorname{Im} z_n \to \operatorname{Im} z \\ |z_n| \to |z| \end{cases}$$

But Arg z may not converge !

§2.6 Convergence of Series

Def: (i) An infinite series $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots$ of cpx numbers converges to the sum $S$ if the sequence of partial sums

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N, \quad N = 1, 2, 3, \cdots$$

converges to $S$, i.e. $\lim_{N\to\infty} S_N = S$.

(ii) When a series doesn't converge, we say that it is diverges.

Thm Suppose that $z_n = x_n + i y_n$ $(n = 1, 2, 3, \cdots)$ and $S = X + iY$.

Then $\sum_{n=1}^{\infty} z_n = S \iff \sum_{n=1}^{\infty} x_n = X$ & $\sum_{n=1}^{\infty} y_n = Y$.

i.e. $$\boxed{\sum_{n=1}^{\infty}(x_n+iy_n) = \left(\sum_{n=1}^{\infty}x_n\right)+i\left(\sum_{n=1}^{\infty}y_n\right)}$$

<u>Cor 1</u> If a series of cpx number converges, then the n-th term converges to zero as $n$ tends to infinity.

<u>Def</u>: A series $\sum_{n=1}^{\infty} z_n$ is said to be <u>absolutely convergent</u> if the (real) series $\sum_{n=1}^{\infty}|z_n|$ converges.

<u>Cor 2</u> The absolute convergence of a series of cpx numbers implies the convergence of that series.

(Pf: Ex!)

<u>Terminology</u>: For a series $\sum_{n=1}^{\infty} z_n \left(= S\right)$ with partial sum

$$S_N = \sum_{n=1}^{N} z_n, \text{ then}$$

$$\rho_N\left(= S - S_N\right) = \sum_{n=N+1}^{\infty} z_n \text{ is called}$$

the <u>remainder after N terms</u> of the series.

And $\sum_{n=1}^{\infty} z_n = S \iff |\rho_N|\left(=|S-S_N|\right) \to 0$ as $N \to +\infty$

eg: $\forall z$ with $|z|<1$, $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$. (Ex!)

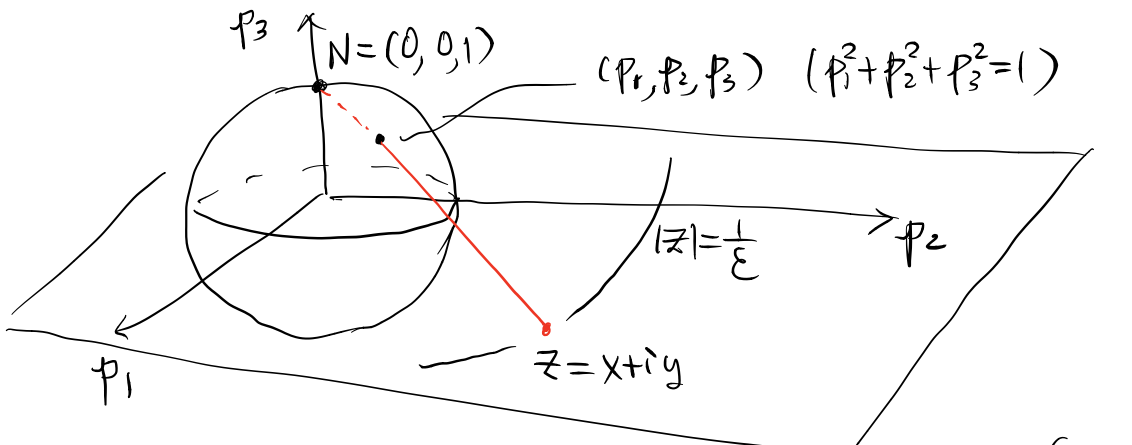
§2.7 Limits involving the point at infinity

Def: The extended complex plane is the union of complex plane $\mathbb{C}$ (= the set of cpx numbers) and the point of infinity $\{\infty\}$.

Notes: (1) We only have one $\infty$.

(Unlike $\mathbb{R}$ with $\pm\infty$, since we don't have a compatible "inequality" on $\mathbb{C}$)

(2) The extended cpx plane $\mathbb{C}\cup\{\infty\}$ can be visualized as a sphere via the stereographic projection:




Then $\begin{cases} z = x+iy = \dfrac{p_1+ip_2}{1-p_3} \\ (p_1,p_2,p_3) = \left(\dfrac{2x}{|z|^2+1}, \dfrac{2y}{|z|^2+1}, \dfrac{|z|^2-1}{|z|^2+1}\right) \end{cases}$ (Ex!)

invertible, 1-1 and onto,

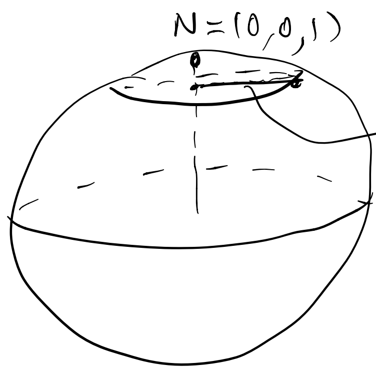
$\therefore \quad S^2 \setminus \{N\} \longleftrightarrow \mathbb{C}$ (1-1 correspondence)

Consider the (very large) circle $|z| = \frac{1}{\varepsilon}$ (for $\varepsilon > 0$ small),

we have

$$(p_1, p_2, p_3) = \left( \frac{2\varepsilon\cos\theta}{1+\varepsilon^2}, \frac{2\varepsilon\sin\theta}{1+\varepsilon^2}, \frac{1-\varepsilon^2}{1+\varepsilon^2} \right)$$

$$\left( \begin{aligned} z &= x+iy \\ &= |z|(\cos\theta + i\sin\theta) \end{aligned} \right)$$




circle with radius $= \frac{2\varepsilon}{1+\varepsilon^2} \to 0$ as $\varepsilon \to 0$.

$\therefore \quad N = (0,0,1) \longleftrightarrow \infty$.

Hence

<u>Def</u>: $\forall \varepsilon > 0$, $\{|z| > \frac{1}{\varepsilon}\}$ is called a <u>neighborhood of $\infty$</u>.

i.e. exterior of the closed disk of radius $\frac{1}{\varepsilon}$ is a nbd. of $\infty$


Note that $\{|z| > \frac{1}{\varepsilon}\} = \{\frac{1}{|z|} < \varepsilon\}$,

Thm: If $z_0$ & $w_0$ are points in the $z$ & $w$-planes
(Def) respectively, then

(1) $$\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$$

(2) $$\lim_{z \to \infty} f(z) = w_0 \iff \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$$

Moreover

(3) $$\lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$$

## §2.8 Continuity

Def: A function $f$ is continuous at a point $z_0$ if

$\lim_{z \to z_0} f(z)$ exists, $f(z_0)$ exists, and

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Thm1 If $f(z) = u(x,y) + i v(x,y)$.

Then $u(x,y)$, $v(x,y)$ are continuous at $(x_0, y_0)$

$\iff$ $f$ is continuous at $z_0 = x_0 + i y_0$

Thm2 Composition of continuous functions is continuous.

Thm3 If $f$ is continuous at $z_0$ and $f(z_0) \neq 0$, then $f(z) \neq 0$ in some nbd. of $z_0$

Thm 4: If $f$ is continuous on a region $R$ that is both closed and bounded, then $\exists M > 0$ such that

$$|f(z)| \leq M, \quad \forall z \in R,$$

where "equality" holds at least for one point.

## §2.9 Derivatives

Def: Let $f$ be a function whose domain of definition contains a nbd. $|z - z_0| < \varepsilon$ of a point $z_0$. The derivative of $f$ at $z_0$ is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

and the function $f$ is said to be (cpx) differentiable at $z_0$ when $f'(z_0)$ exists.

Usual notations: $\begin{cases} \Delta z = z - z_0 \\ \Delta w = f(z_0 + \Delta z) - f(z_0) \end{cases}$ $\quad (w = f(z))$

We often drop the subscript on $z$ and write

$$\Delta w = f(z + \Delta z) - f(z).$$

Then $\quad f'(z) = \dfrac{dw}{dz} = \lim\limits_{\Delta z \to 0} \dfrac{\Delta w}{\Delta z}.$

eg: Let $w = f(z) = \bar{z} = x - iy$.

i.e. $\begin{cases} u = x \\ v = -y \end{cases}$

$u, v$ are clearly (real) differentiable.

But $\dfrac{\Delta w}{\Delta z} = \dfrac{f(z+\Delta z) - f(z)}{\Delta z} = \dfrac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z}$

$= \dfrac{\overline{(\Delta z)}}{\Delta z}$ limit doesn't exist (as $\Delta z \to 0$)

$\therefore$ $f(z) = \bar{z}$ is not (cpx) differentiable.

## §2.10 Rules and properties of differentiation

(1) Diffentiability $\Rightarrow$ Continuity

(but Continuity $\not\Rightarrow$ Differentiability)

(2) If derivatives of $f(z)$ and $g(z)$ exist at $z$,

then (a) $\dfrac{d}{dz} c = 0$, for const. $c$.

(b) $\forall$ integer $n \geq 1$, $\dfrac{d}{dz} z^n = n z^{n-1}$.

(c) $\dfrac{d}{dz}(f \pm g) = \dfrac{df}{dz} \pm \dfrac{dg}{dz}$

(d) $\dfrac{d}{dz}(fg) = f(z)\dfrac{dg}{dz} + \dfrac{df}{dz} g(z)$

(e) If $g(z) \neq 0$, then $\dfrac{d}{dz}\left(\dfrac{f}{g}\right) = \dfrac{g\frac{df}{dz} - f\frac{dg}{dz}}{g^2}$.

(3) Chain Rule: If $f$ has derivatives at $z_0$, $g$ has derivatives at $f(z_0)$. Then $F(z)=g(f(z))$ has derivatives at $z_0$ and

$$\boxed{F'(z_0)=g'(f(z_0))f'(z_0)}$$

i.e. $$\frac{dF}{dz} = \frac{dg}{dw}\frac{dw}{df}.$$

## § 2.11 Cauchy-Riemann Equations

Thm 1 Suppose that $f(z)=u(x,y)+iv(x,y)$

Then $f'(z)$ exists at a point $z_0=x_0+iy_0$ if and only if the mapping $F\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}u(x,y)\\v(x,y)\end{pmatrix}$ is (real) differentiable at $\begin{pmatrix}x_0\\y_0\end{pmatrix}$ and satisfies the Cauchy-Riemann equations

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{at } (x_0,y_0).$$

Moreover

$$f'(z_0)= u_x(x_0,y_0)+iv_x(x_0,y_0).$$

Pf: $f'(z)$ exists at $z_0=x_0+iy_0$

$$\Leftrightarrow \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = f'(z_0)$$

$$\Leftrightarrow \lim_{|\Delta z| \to 0} \left| \frac{f(z+\Delta z)-f(z) - f'(z_0)\Delta z}{\Delta z} \right| = 0$$

Let $\Delta z = \Delta x + i \Delta y$ and $f'(z_0) = \alpha + i\beta$ $(\alpha, \beta \in \mathbb{R})$

Then $f'(z_0)\Delta z = (\alpha + i\beta)(\Delta x + i\Delta y)$

$$= (\alpha\Delta x - \beta\Delta y) + i(\beta\Delta x + \alpha\Delta y).$$

$\therefore$ $f'(z)$ exist at $z_0 = x_0 + i y_0$

$$\Leftrightarrow \lim_{\sqrt{\Delta x^2+\Delta y^2} \to 0} \frac{\left| \left[u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) - (\alpha\Delta x - \beta\Delta y)\right] + i\left[v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0) - (\beta\Delta x + \alpha\Delta y)\right] \right|}{\sqrt{\Delta x^2+\Delta y^2}} = 0$$

$$\Leftrightarrow \lim_{\sqrt{\Delta x^2+\Delta y^2} \to 0} \frac{1}{\sqrt{\Delta x^2+\Delta y^2}} \left| \begin{pmatrix} u(x_0+\Delta x, y_0+\Delta y) \\ v(x_0+\Delta x, y_0+\Delta y) \end{pmatrix} - \begin{pmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{pmatrix} - \begin{pmatrix} \alpha\Delta x - \beta\Delta y \\ \beta\Delta x + \alpha\Delta y \end{pmatrix} \right| = 0$$

$$\Leftrightarrow \lim_{\sqrt{\Delta x^2+\Delta y^2} \to 0} \frac{1}{\sqrt{\Delta x^2+\Delta y^2}} \left| F\begin{pmatrix} x_0+\Delta x \\ y_0+\Delta y \end{pmatrix} - F\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \right| = 0$$

$\Leftrightarrow$ $F\begin{pmatrix} x \\ y \end{pmatrix}$ is differentiable at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ with Jacobi matrix (differential) $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$.

$\Leftrightarrow$ $F\begin{pmatrix} x \\ y \end{pmatrix}$ is differentiable at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ with Cauchy-Riemann equations $\begin{cases} u_x = v_y & (= \alpha = \operatorname{Re} f'(z_0)) \\ u_y = -v_x & (= -\beta = -\operatorname{Im} f'(z_0)) \end{cases}$.

#

Thm 2 : Suppose that $f(z) = u(x,y) + i v(x,y)$ and $f'(z)$ exists at a point $z_0 = x_0 + i y_0$. Then the partial derivatives $u_x, u_y, v_x, v_y$ exist at $(x_0, y_0)$ and satisfy the Cauchy-Riemann equations $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ at $(x_0, y_0)$.

(Pf: Follows immediately from Thm 1.)

eg3 : $f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$

For $z \neq 0$, $f(z) = \dfrac{x^3 - 3xy^2}{x^2+y^2} + i \dfrac{(-3x^2y + y^3)}{x^2+y^2}$

i.e., $u(x,y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

$$v(x,y) = \begin{cases} \dfrac{y^3-3x^2y}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

Then

$$\begin{cases} u_x(0,0) = \lim\limits_{\Delta x\to 0} \dfrac{u(0+\Delta x,0)-u(0,0)}{\Delta x} = 1 \\ u_y(0,0) = \quad \cdots \quad = 0 \\ v_x(0,0) = \quad \cdots \quad = 0 \\ v_y(0,0) = \quad \cdots \quad = 1 \end{cases}$$

(Ex.)

$$\Rightarrow \begin{cases} u_x(0,0) = v_y(0,0) \\ u_y(0,0) = -v_x(0,0) \end{cases}$$

$\therefore$ CR equations satisfied at $(0,0)$.

However
$$\frac{f(0+\Delta z)-f(0)}{\Delta z} = \frac{\dfrac{(\overline{\Delta z})^2}{\Delta z} - 0}{\Delta z} = \left(\frac{\overline{\Delta z}}{\Delta z}\right)^2$$

limit doesn't exist as $\Delta z \to 0$.

$\therefore$ $f(z)$ is <u>not</u> (cpx) differentiable at $(0,0)$.

## § 2.12 Sufficient conditions for Differentiability

<u>Thm</u>: Let $f(z) = u(x,y)+iv(x,y)$ defined throughout some $\varepsilon$-nbd $|z-z_0|<\varepsilon$ of $z_0 = x_0+iy_0$, and

(a) $u_x, u_y, v_x, v_y$ exist <u>everywhere</u> in $|z-z_0|<\varepsilon$.

(b) $u_x, u_y, v_x, v_y$ are continuous at $(x_0, y_0)$ and satisfy $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ at $(x_0, y_0)$

Then $f'(z_0)$ exists and $f'(z_0) = (u_x + i v_x)_{(x_0, y_0)}$.

Pf: Conditions $\Rightarrow$ $u, v$ differentiable at $(x_0, y_0)$ and satisfy CR equations.

By Thm 1 in §2.11, $f'(z_0)$ exists & $f'(z_0) = (u_x + i v_x)_{(x_0, y_0)}$

#

§ 2.13 Polar coordinates

Thm Let $f(z) = u(r, \theta) + i v(r, \theta)$ be defined in some $\varepsilon$-nbd of a nonzero point $z_0 = r_0 e^{i\theta_0}$, and suppose that

(a) $u_r, u_\theta, v_r, v_\theta$ exist everywhere in the $\varepsilon$-nbd.

(b) $u_r, u_\theta, v_r, v_\theta$ continuous at $(r_0, \theta_0)$ satisfying

$$\begin{cases} u_r = \frac{1}{r} v_\theta \\ \frac{1}{r} u_\theta = - v_r \end{cases}$$ the Polar Form of CR equations at $(r_0, \theta_0)$

Then $f'(z_0)$ exists and

$$f'(z_0) = e^{-i\theta_0} \left( u_r(r_0, \theta_0) + i v_r(r_0, \theta_0) \right).$$