Ch1 Complex Number

Standard notations:

$$\begin{cases} \mathbb{N} = \{0, 1, 2, 3, \cdots\} & \text{set of natural numbers} \\ \mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\} & \text{set of integers.} \\ \mathbb{Q} = \text{set of rational numbers} \\ \mathbb{R} = \text{set of real numbers} \end{cases}$$

§1.1 Sums & Product

The set of complex numbers $\mathbb{C} = \{z = x + iy : x, y \in \mathbb{R}\}$

For $z = x + iy$, $\begin{cases} x = \operatorname{Re} z & \text{real part of } z \\ y = \operatorname{Im} z & \text{imaginary part of } z \end{cases}$

$$\begin{cases} z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \\ z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2) \end{cases}$$

$$\left(\Rightarrow \quad i^2 + 1 = 0 \quad (\text{check!}) \right)$$

Basic Algebraic Properties

$$\begin{cases} z_1 + z_2 = z_2 + z_1 \\ (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \\ \exists\, 0 \text{ s.t. } z + 0 = z, \ \forall z \\ \forall z, \ \exists\, -z \text{ s.t. } z + (-z) = 0 \end{cases}$$

$$\begin{cases} z_1 z_2 = z_2 z_1 \\ (z_1 z_2) z_3 = z_1 (z_2 z_3) \\ \exists\, 1 \text{ s.t. } z \cdot 1 = z \\ \forall z \neq 0, \exists\, z^{-1} \text{ s.t. } z z^{-1} = 1 \end{cases}$$

- $z(z_1 + z_2) = z z_1 + z z_2$

Notes: (1) $\forall n \in \mathbb{Z}$, $z^n \overset{\text{def}}{=} \underbrace{z \cdots z}_{n}$, $z^0 = 1$.

(2) $z_1 z_2 = 0 \Rightarrow z_1 = 0$ or $z_2 = 0$

(3) subtraction $z_1 - z_2 \overset{\text{def}}{=} z_1 + (-z_2)$

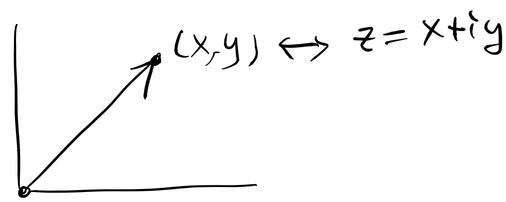
Division $\dfrac{z_1}{z_2} \overset{\text{def}}{=} z_1 z_2^{-1}$ (for $z_2 \neq 0$)

(4) $-z = (-1) z$

Vectors & Moduli

(1) $\forall\ z = x + iy \in \mathbb{C} \longleftrightarrow (x, y) \in \mathbb{R}^2$

cpx number   plane vector



$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ cpx addition

$\updownarrow$ $\updownarrow$

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2)$ vector addition

(2) But cpx multiplication is neither the scalar ~~product~~ multiplication nor vector product (as in vector analysis)

(i) scalar ~~product~~ multiplication $\alpha(x,y) = (\alpha x, \alpha y)$, $\alpha \in \mathbb{R}$ defined only for $\alpha \in \mathbb{R}$

Complex multiplication is an extension to allow $\alpha \in \mathbb{C}$.

(ii) vector product takes 2 plane vectors to a vector perpendicular to the plane, hence not in the plane $\mathbb{R}^2$.

Def: The modulus (or absolute value) of $z = x+iy$ is defined by

$$|z| = \sqrt{x^2+y^2}$$

= length of the vector (x,y)

= distance between (x,y) & $(0,0)$.

Notes: (1) The inequality $z_1 < z_2$ is not defined for cpx numbers. Therefore $z_1 < z_2$ is meaningless unless $z_1, z_2 \in \mathbb{R}$. However $|z_1| < |z_2|$ is meaningful.

(2) $\begin{cases} \operatorname{Re} z \leq |\operatorname{Re} z| \leq |z| \\ \operatorname{Im} z \leq |\operatorname{Im} z| \leq |z| \end{cases}$

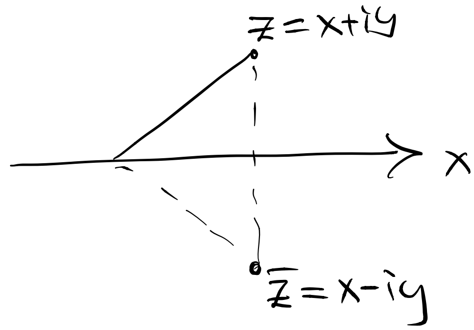
(3) <u>Triangle inequality</u> $|z_1 + z_2| \leq |z_1| + |z_2|$

and hence $\big|\,|z_1| - |z_2|\,\big| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$ (Ex).

§1.2 Complex Conjugate

<u>Def</u>: The <u>complex conjugate</u> (or simply <u>conjugate</u>) of $z = x + iy$ is

$$\boxed{\bar{z} = x - iy}$$



$\bar{z}$ is represented by the <u>reflection in real axis</u>.

$$\begin{cases} \overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2 \\ \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \end{cases}, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

$$\begin{cases} \operatorname{Re} z = \dfrac{z + \bar{z}}{2} \\ \operatorname{Im} z = \dfrac{z - \bar{z}}{2i} \end{cases}$$

- $z\bar{z} = |z|^2$

§1.3 Exponential Form

Polar coordinate (r, θ) for (x, y):

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad (r \geq 0,\ \theta \in \mathbb{R})$$

$\Rightarrow \quad r = |z|$

Notes (i) θ is undefined for $z = 0$

(ii) θ is only defined up to $2k\pi$, $k \in \mathbb{Z}$

i.e. $\forall\ \theta = 2k\pi,\ k \in \mathbb{Z}$

$$z = r(\cos\theta + i\sin\theta)$$
$$= r[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)].$$

Definitions

(1) Each value of θ s.t. $z = |z|(\cos\theta + i\sin\theta)$ is called an argument of z.

(2) $\boxed{\arg z = \text{set of all arguments of } z}$

(3) The principal value of $\arg z$, a principal argument of z, denoted by $\operatorname{Arg} z$ is the value

$$\Theta \in \arg z \text{ such that } -\pi < \Theta \leq \pi$$

Then $\begin{cases} \arg z = \{ \operatorname{Arg} z + 2k\pi : k \in \mathbb{Z} \} \text{ is a set} \\ \qquad = \operatorname{Arg} z + 2k\pi, \; k \in \mathbb{Z} \text{ (for simplicity)} \\ \operatorname{Arg} z \in (-\pi, \pi] \end{cases}$

Notation:

Define $\underline{e^{i\theta} = \cos\theta + i\sin\theta, \quad \forall \theta \in \mathbb{R}}$

(Euler formula)

Then $z = r(\cos\theta + i\sin\theta) = \underline{re^{i\theta}}$ (exponential form of z)

$\left(= |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta} = |z|e^{i\arg z} \right)$

eg: $z = z_0 + Re^{i\theta}$, $\theta \in (-\pi, \pi]$, represents a circle of radius R centered at z_0.

§1.4 Products & Powers in Exponential form

$$\underline{e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}} \quad \text{(By compound angle formula)}$$

For $z_1 = r_1 e^{i\theta_1}$ & $z_2 = r_2 e^{i\theta_2}$

$$\begin{cases} z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \\ \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \qquad (r_2 \neq 0) \\ z_1^n = r_1^n e^{in\theta_1} \end{cases}$$

deMeivre's famula

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta} \quad (\forall n \in \mathbb{Z}.)$$

$$\boxed{\arg(z_1 z_2) = \arg z_1 + \arg z_2} \quad (\text{as sets})$$

Roots of complex numbers

For $z_0 = r_0 e^{i\theta_0}$ $(\neq 0)$

Then
$$\boxed{c_k = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, 2, \cdots, n-1.} \quad (*)$$

are all the distinct n-roots of z_0

Notations (1) $z_0^{\frac{1}{n}}$ denote set of all n-roots of z_0

$$= \{c_0, c_1, \cdots, c_{n-1}\} \quad (c_k \text{ as in } (*))$$

In this notation
$$r_0^{\frac{1}{n}} = \{\sqrt[n]{r_0}\, e^{i\frac{2\cdot 0\cdot\pi}{n}}, \sqrt[n]{r_0}\, e^{i\frac{2\cdot 1\cdot\pi}{n}}, \ldots, \sqrt[n]{r_0}\, e^{i\frac{2(n-1)\pi}{n}}\}$$
$$= \{\sqrt[n]{r_0}\, e^{i\frac{2k\pi}{n}}, \ k = 0, 1, \cdots, n-1\}$$

$\therefore$ $r_0^{\frac{1}{n}}$ is a set, but $\sqrt[n]{r_0}$ is a positive real number

s.t. $(\sqrt[n]{r_0})^n = r_0$ $(r_0 > 0)$

(2) Principal n-root :

If $z_0 = r_0 e^{i\theta_0}$ with $\theta_0 = \mathrm{Arg}\, z \in (-\pi, \pi]$

then $\underline{c_0 = \sqrt[n]{r_0}\, e^{i \frac{\operatorname{Arg} z_0}{n}}}$ is called the $\underline{\text{Principal } n\text{-root}}$ of z_0.

(3) $\omega_n \overset{\text{denote}}{=\!=} e^{i\frac{2\pi}{n}}$ satisfies

$$\begin{cases} (\omega_n)^k = e^{i\frac{2k\pi}{n}} \\ \omega_n^n = 1 \end{cases}$$

$\therefore$ ω_n is called the $\underline{n\text{-root of unity}}$

(ω_n^k are called the $\underline{n\text{-roots of unity}}$).

With this notation $z_0^{\frac{1}{n}} = \{ c_0\, \omega_n^k ,\ k = 0, 1, 2, \cdots, n-1 \}$

where c_0 = principal n-root of z_0.

i.e. $z_0^{\frac{1}{n}} = \{$ "principal n-root" $\times$ "n-roots of unity" $\}$

§1.5 Regions in the complex plane

Def: (1) $B_\varepsilon(z_0) = \{ z \in \mathbb{C} : |z - z_0| < \varepsilon \}$ is called the $\underline{\varepsilon\text{-neighborhood}}$ (ε-nbd) of the point z_0

(2) $B_\varepsilon(z_0) \setminus \{z_0\} = \{ z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon \}$ is called the $\underline{\text{deleted } \varepsilon\text{-nbd}}$.

Terminology in topology:

interior point, exterior point, interior of a set, exterior of a set, boundary of a set, boundary point, open set, closed set, closure of a set, connected set, bounded set, unbounded set, and accummulation point of a set

are the same as in $\mathbb{R}^2$.

Ch.2 Analytic Functions

§2.1 Functions and Mappings

Let S be a set of cpx numbers.

Def: (1) A function f defined on S is a rule that assigns to each $z \in S$, a complex number w, denoted by

$$w = f(z) \in \mathbb{C}.$$

(2) The cpx number $w = f(z)$ is called the value of f at z.

(3) S is called the domain (of definition) of f

Convention: When the domain of f is not mentioned we agree that the largest possible set is to be taken.

If $z = x+iy$ and $w = f(z) = u+iv$,

i.e. $\quad u+iv = f(z) = f(x+iy)$

$\Rightarrow \quad \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$ are real-valued functions of 2-variables (x,y).

and write $\underline{f(z) = u(x,y) + i\,v(x,y)}$.

eg: $f(z) = z^2 = (x+iy)^2 = (x^2-y^2) + i(2xy)$.

$$\therefore \begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}.$$

<u>Terminology</u>:

(1) $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ with $a_n \neq 0$ is <u>a polynomial of degree n</u>.

(2) Quotient $\dfrac{P(z)}{Q(z)}$ of polynomials $P(z)$ & $Q(z)$ are called <u>rational functions</u> (defined at z with $Q(z) \neq 0$)

<u>Polar coordinates</u> $z = x + iy = re^{i\theta}$

$$\begin{cases} u = u(r,\theta) \\ v = v(r,\theta) \end{cases}$$

and we may write

$$\boxed{f(z) = u(r,\theta) + i\,v(r,\theta)} \quad \text{for } z = re^{i\theta}$$

eg $w = f(z) = z^2$ for $z = re^{i\theta}$

$$= (re^{i\theta})^2 = r^2 e^{i2\theta} = r^2(\cos 2\theta + i \sin 2\theta)$$

$$\Rightarrow \quad u = r^2 \cos 2\theta \;\; \& \;\; v = r^2 \sin 2\theta.$$

Multiple-valued functions : assigns more than one value to a point z in the domain of definition.

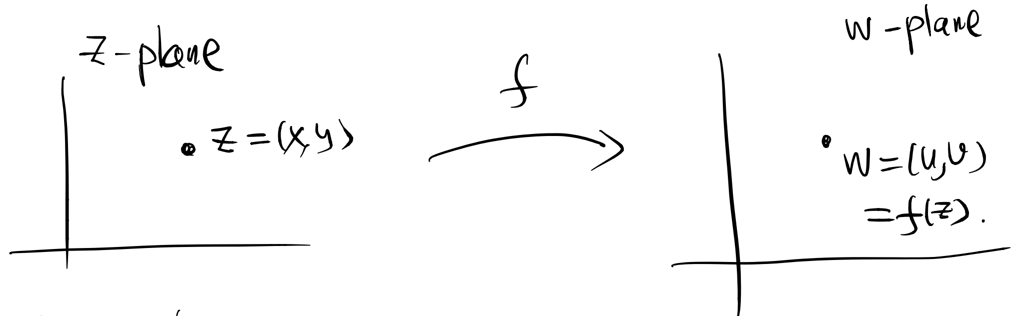
eg: $z \longmapsto z^{\frac{1}{n}} = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}$, $k = 0, 1, \cdots, n-1$

is a multiple-valued function for $n \geq 2$.

Terminology

(1) Mapping or transformation

when a function f is thought of correspondence between points $z = (x, y)$ & $w = (u, v)$:



(2) The point $w = (u, v) = f(z)$ is called the image of the point $z = (x, y)$ under the mapping (transformation) $w = f(z)$.

(3) Range of $f = \{ w : w = f(z),\ \forall z \in S \}$

(4) Inverse image (preimage) of a point w_0 is

$$f^{-1}(w_0) \overset{\text{def}}{=\!=} \{ z \in S : f(z) = w_0 \}$$

<u>egs</u>: (1) <u>translation</u> $w = f(z) = z + b$, where b is a fixed complex number.

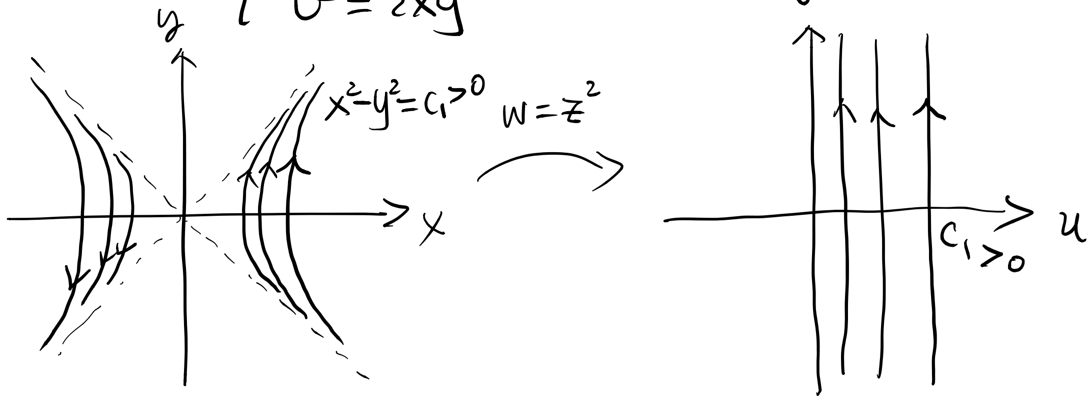
(2) <u>rotation</u> $w = f(z) = e^{j\theta} z$, where θ is a fixed <u>real</u> number,

(3) <u>reflection</u> $w = f(z) = \bar{z}$.

§2.2 <u>The Mapping $w = z^2$</u>

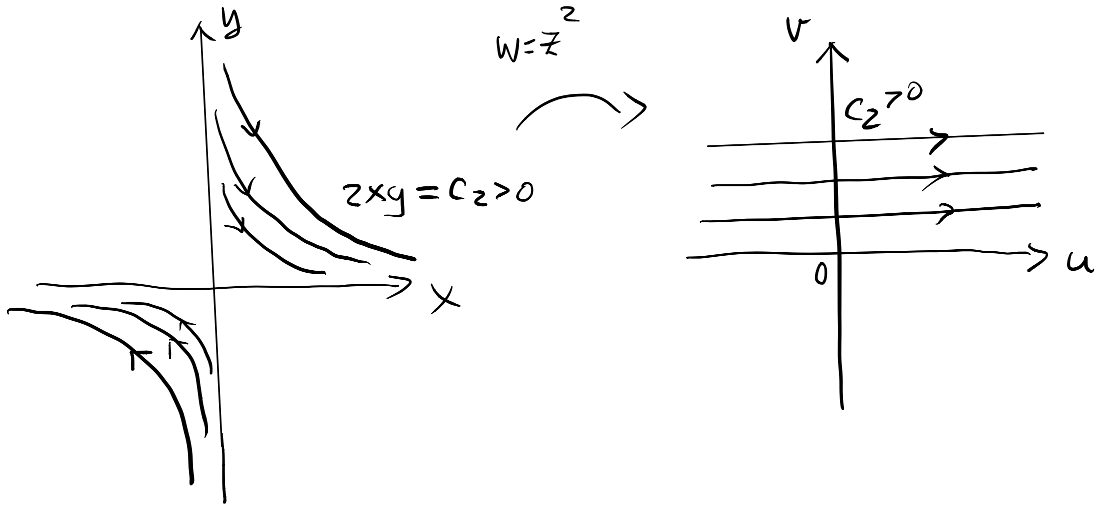
The $w = z^2$ can be thought of the transformation

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$



Ex: what happen for $c_1 = 0$ & $c_1 < 0$?

Similarly, we can consider



Ex: what happen for $c_2 = 0$ & $c_2 < 0$.

In polar coordinate, i.e. exponential form for $w = z^2$:

$$w = (re^{i\theta})^2 = r^2 e^{2i\theta}.$$

$$|w| = r^2 = |z|^2 \Rightarrow$$