THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 9 Solutions

Problems that may be demonstrated in class :

Q1. Compute the following definite integrals:
(a) $\int_0^3 (x-1)(x+2)\,dx$ (b) $\int_0^3 x\lfloor x\rfloor dx$ (c) $\int_0^{\pi/2} \sin x\cos^4 x dx$ (d) $\int_1^e \frac{\ln x}{x}dx$

(e) $\int_2^4 \sqrt{16-x^2}\,dx$

Here $\lfloor x\rfloor$ is the greatest integer that is less than or equal to x.

Q2. (a) Let f be continuous on $[0,1]$. Prove that $\int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx$
(b) Evaluate $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$

Q3. Suppose $a>0$ and that f is continous on $\mathbb{R}$.
(a) If f is even, show that $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$
(b) If f is odd, show that $\int_{-a}^a f(x)dx = 0$
(c) Show that $\int_{-a}^a f(x^2)dx = 2\int_0^a f(x^2)dx$

Q4. Suppose that f is continuous on $[a,b]$, $f(x)\geq 0$ for all $x\in[a,b]$ and $\int_a^b f(x)dx = 0$. Prove that $f(x)=0$ for all $x\in[a,b]$.

Q5. Compute $F'(x)$ if $F(x)$ equals
(a) $\int_1^x e^{t^2}\,dt$ (b) $\int_1^{x^2} e^{t^2}\,dt$ (c) $\int_0^{3x}\tan(t^2)dt$ (d) $\int_{-x}^{x^2+3}\arctan t\,dt$

Q6. Let f be a continuous function on $\mathbb{R}$ and F be a primitive function of f. Let $a,b\in\mathbb{R}$ and $a<b$. Show that there exists $c\in(a,b)$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Solution

Q1. (a) $\int_0^3 (x-1)(x+2)dx = \int_0^3 x^2+x-2dx = \left(\frac{x^3}{3}+\frac{x^2}{2}-2x\right)_0^3 = \frac{15}{2}$

(b) Note that

$$x\lfloor x\rfloor = \begin{cases} 0 & \text{if } 0\leq x<1 \\ x & \text{if } 1\leq x<2 \\ 2x & \text{if } 2\leq x<3 \\ 9 & \text{if } x=3 \end{cases}$$

Hence $\int_0^3 x\lfloor x\rfloor dx = \int_0^1 0dx + \int_1^2 xdx + \int_2^3 2xdx = \frac{13}{2}$

(c) $\int_0^{\pi/2}\sin x\cos^4 xdx = -\int_0^{\pi/2}\cos^4 x\,d(\cos x) = \left(\frac{-\cos^5 x}{5}\right)_0^{\pi/2} = \frac{1}{5}$

(d) $\int_1^e \frac{\ln x}{x}dx = \int_1^e \ln x\,d(\ln x) = \left(\frac{\ln^2 x}{2}\right)_1^e = \frac{1}{2}$

(e) Let $x=4\sin\theta$, we have $dx = 4\cos\theta d\theta$. When $x=2, \theta=\frac{\pi}{6}$. When $x=4,\theta=\pi$. Hence

$$\int_2^4 \sqrt{16-x^2}\,dx = \int_{\frac{\pi}{6}}^{\pi} 4\cos\theta\cdot 4\cos\theta\,d\theta = 16\int_{\frac{\pi}{6}}^{\pi}\frac{\cos 2\theta+1}{2}d\theta = \frac{8\pi}{3}-2\sqrt{3}$$

Q2. (a) Let $I = \int_0^\pi xf(\sin x)dx$ and $y = \pi - x$, we have

$$I = \int_\pi^0 (\pi - y)f(\sin(\pi - y))(-dy) = \int_0^\pi (\pi - y)f(\sin y)dy = \pi \int_0^\pi f(\sin y)dy - I$$

Hence

$$2I = \pi \int_0^\pi f(\sin x)dx \quad \text{and} \quad I = \frac{\pi}{2}\int_0^\pi f(\sin x)dx$$

(b) Note that $\frac{\sin x}{1+\cos^2 x} = f(\sin x)$, where $f(x) = \frac{x}{2-x^2}$. By (a),

$$\begin{aligned}\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx &= \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx \\ &= \frac{-\pi}{2}\int_0^\pi \frac{d(\cos x)}{1+\cos^2 x} \\ &= \frac{-\pi}{2}\left(\arctan(\cos x)\right)_0^\pi \\ &= \frac{\pi^2}{4}\end{aligned}$$

Q3. (a) By letting $y = -x$ we obtain $\int_{-a}^0 f(x)dx = \int_0^a f(y)dy$, since $f(x) = -f(x)$. This completes our proof.
(b) Again, let $y = -x$, we have $\int_{-a}^0 f(x)dx = -\int_0^a f(y)dy$ and thus $\int_{-a}^a f(x)dx = 0$
(c) Let $g(x) = f(x^2)$. Since g is even, by (a) we have $\int_{-a}^a f(x^2)dx = 2\int_0^a f(x^2)dx$

Q4. Suppose there exists $c \in (a,b)$ such that $f(c) > 0$. Then by continuity, there exists an interval (p,q) containing c such that $f(x) > 0$ for all $x \in (p,q)$. Then we have

$$\int_a^b f(x)dx \geq \int_p^q f(x)dx > 0 \quad (\because f(x) > 0\,\forall\, x \in (p,q))$$

Then we obtain a contradiction, and hence f must be identically zero.

Q5. (a) $F'(x) = e^{x^2}$
(b) $F'(x) = f(x^2)\dfrac{dx^2}{dx} = 2xe^{x^4}$
(c) $F'(x) = \tan(9x^2)\dfrac{d(3x)}{dx} = 3\tan(9x^2)$
(d)

$$\begin{aligned}F'(x) &= 2x\arctan(x^2+3) - (-1)\arctan(-x) \\ &= 2x\arctan(x^2+3) + \arctan(-x) \\ &= 2x\arctan(x^2+3) - \arctan(x)\end{aligned}$$

Since arctan is an odd function.

Q6. Since F is differentiable, by mean value theorem, there exists $c \in (a,b)$ such that

$$F'(c) = \frac{F(b) - F(a)}{b - a}$$

Hence

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$