MAT 4250: Lecture 1

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Chapter 1: Impartial Combinatorial Games

Combinatorial games

Combinatorial games are two-person games with perfect information and no chance moves, and with a win-or-lose outcome.

Such a game is determined by a set of positions (including initial positions), and the players.

Play moves from one position to another, with the players alternating moves, until a terminal position is reached.

A terminal position is one from which no moves are possible.

Then one player is declared the winner and the other the loser.

Impartial games: the set of moves available from any given position is the same for both players.

Partizan games: player has a different set of possible moves from a given position.

We treat only impartial games.

A simple take-away game

Rules for this simple impartial combinatorial game:

1. There are two players, labelled I and II.
2. There is a pile of 21 chips on the table.
3. A move consists of removing $1, 2$ or 3 chips from the pile.
4. Players alternate moves with Player I starting.
5. The player that removes the last chip wins.

Questions: How to analyze this game? Can one of the players force a win in this game? Which player would you rather be, the player who starts or the player who goes second? What is a good strategy?

We use **backward induction** to analyze this game.

- If there are just $1, 2$ or 3 chips left, the next player wins.
- If there are 4 chips left, then the next player must leave $1, 2$ or 3 chips and his opponent will win. Hence, 4 chips left is a loss for the next player and is a win for the previous player.
- With $5, 6$ or 7 chips left, the next player can win by moving to the position with 4 chips left.
- With 8 chips left, the next player to move must leave $5, 6$ or 7 chips. So, the previous player wins.

We see that $0, 4, 8, 12, 16, 20$ are target positions, we would like to move into them.

The first player wins by removing one chip and leave 20 chips.

Precise definition of combinatorial games

Combinatorial game is a game that satisfies the conditions:

1. There are two players.
2. There is a finite set of possible positions.
3. The rules of the game specify for both players and each position which moves to other positions are legal moves. If the rules make no distinction between the players, the game is called **impartial**. Otherwise, the game is called **partizan**.
4. The players alternate moving.
5. The game ends when a position is reached from which no moves are possible. Under the **normal play rule**, the last player to move wins.
6. The game ends in a finite number of moves.

Remarks:

1. Under the **misère play rule**, the last player to move loses.
2. If a game never ends, it is declared a draw. We can always add an **ending condition** to eliminate this possibility.
3. No random move such as rolling of a dice is allowed.
4. A combinatorial game is a game with perfect information. Simultaneous moves and hidden moves are not allowed.

P-position and N-position

Recall that, in the above take-away game, we see that $0, 4, 8, \cdots$ are positions that are winning for the **P**revious player and that $1, 2, 3, 5, \cdots$ are positions that are winning for the **N**ext player.

$0, 4, 8, \cdots$ are called P-positions and $1, 2, 3, 5, \cdots$ are called N-positions.

P-positions are positions that are winning for the previous player and N-positions are positions that are winning for the next player.

In impartial combinatorial games, one can find in principle which positions are P-positions and which are N-positions.

We say a position is a **terminal position** if no moves from it are possible.

Finding P- and N-positions

The method is very similar to the way we solve the take-away game.

1. Label every terminal position as a P-position.
2. Label every position that can reach a labelled P-position in one move as an N-position.
3. Find those positions whose only moves are to labelled N-positions. And label such positions as P-positions.
4. If no new P-positions were found in step 3, stop. Otherwise return to step 2.

The strategy of moving to P-positions wins. From a P-position, your opponent can only move to N-position (3). Then you can move back to a P-position. Eventually the game ends at a terminal position (which is a P-position).

Characteristic property: (Under the normal play rule)

P-positions and N-positions are define recursively by the following

1. All terminal positions are P-positions.
2. From every N-position, there is at least one move to a P-position.
3. From every P-position, every move is to an N-position.

Subtraction games

Consider a class of combinatorial games that contains the above take-away game.

Let S be a set of positive integers.

The subtraction game with subtraction set S is played as follows. From a pile with a large number, say n, of chips, two players alternate moves. A move consists of removing s chips from the pile where $s \in S$. Last player to move wins.

The above take-away game is a subtraction game with $S = \{1, 2, 3\}$.

An example: take $S = \{1, 3, 4\}$.

There is exactly one terminal position, namely 0. Thus it is a P-position.

Then $1, 3, 4$ are N-positions, since they can move to 0.

2 must be a P-position because the only legal move from 2 is to 1.

$5, 6$ must be N-positions since they can be moved to 2.

7 must be a P-position since the only moves from 7 are to $6, 4, 3$, which are N-positions.

Similarly, $8, 10, 11$ are N-positions, 9 is a P-position, $12, 13$ are N-positions and 14 is a P-position.

Now repeat inductively, we see that P-positions are $\{0, 2, 7, 9, 14, 16, \cdots\}$ and N-positions are $\{1, 3, 4, 5, 6, 8, 10, 11, 12, 13, 15, \cdots\}$.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
po	P	N	P	N	N	N	N	P	N	P	N	N	N	N

Note that the pattern PNPNNNN of length repeats forever.

Q: who wins the game with 100 chips, the first or the second player?

The P-positions are the numbers equal to 0 or 2 modulus 7. Since 100 has remainder 2 when divided by 7, 100 is a P-position. Hence the second player will win with optimal play.

The game of Nim

The most famous take-away game is the game of Nim.

There are 3 piles of chips containing x_1, x_2 and x_3 chips. Two players take turn moving. Each move consists of selecting one of the piles and removing chips from it. You cannot remove chips from more than one pile, but from the pile you selected you may remove as many as you want.

The winner is the one who removes the last chip.

You can play at http://www.dotsphinx.com/nim

Preliminary analysis

Exactly one terminal position $(0, 0, 0)$, which is a P-position.

Any position with one pile, say $(0, 0, x)$ with $x > 0$, is a N-position because you can win by removing all chips.

Consider two non-empty piles. We see that the P-positions are those for which the two piles have an equal number of chips, e.g. $(0, 1, 1)$. This is because your opponent must move to a position with an unequal number of chips, and then you can return to a position with equal number of chips.

Consider all 3 piles are non-empty. Clearly $(1, 1, 2), (1, 1, 3), (1, 2, 2)$ are N-positions because they can move to $(1, 1, 0), (1, 1, 0), (0, 2, 2)$. Then we see that $(1, 2, 3)$ is a P-position because it can only be moved to N-positions.

How to generalize?

Nim-sum

Every non-negative integer x has a unique base 2 representation of the form $x = x_m 2^m + x_{m-1} 2^{m-1} + \cdots + x_1 2 + x_0$ for some m, where each x_i is either 0 or 1. We use $(x_m x_{m-1} \cdots x_1 x_0)_2$ to denote this representation.

Ex: $22 = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2 + 0 \cdot 1 = (10110)_2$.

Definition: The nim-sum of $(x_m \cdots x_0)_2$ and $(y_m \cdots y_0)_2$ is $(z_m \cdots z_0)_2$, and write

$$(x_m \cdots x_0)_2 \oplus (y_m \cdots y_0)_2 = (z_m \cdots z_0)_2$$

where $z_k = x_k + y_k \pmod 2$.

It is the component-wise addition modulo 2 for the base 2 representation.

Example: $22 \oplus 51 = 37$.

Note that $22 = (010110)_2$ and $51 = (110011)_2$.

Component-wise addition modulo 2 gives $(100101)_2$, which is 37.

Remark:

Nim-sum is associative, i.e., $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

Nim-sum is commutative, i.e., $x \oplus y = y \oplus x$.

0 is an identity, i.e., $0 \oplus x = x$.

Every number is its own negative, i.e., $x \oplus x = 0$.

Cancellation law holds, i.e., if $x \oplus y = x \oplus z$, then $y = z$.

Question: what nim-sum have to do with playing the game of Nim?

Theorem: A position, (x_1, x_2, x_3), in the game of Nim is a P-position if and only if the nim-sum of its components is zero, $x_1 \oplus x_2 \oplus x_3 = 0$.

Example: Consider the position $(13, 12, 8)$. Is this a P-position? If not, what is a winning move?

Note that $13 = (1101)_2$, $12 = (1100)_2$ and $8 = (1000)_2$.

The nim-sum is $(1001)_2 = 9$. Thus it is a N-position.

How do we find a winning move? We need to find a move to a P-position. We need to move to a position such that there are even number of 1 in each component.

Simply take away 9 chips from the first pile.

<u>Proof of theorem</u>: need to check the 3 conditions.

Let $\mathcal{P}$ be the set of positions with nim-sum zero and $\mathcal{N}$ be the complement (with positive nim-sum).

1. *All terminal positions are in $\mathcal{P}$.* The only terminal position is $(0, 0, 0)$, so it is in $\mathcal{P}$.

2. *From each position in $\mathcal{N}$, there is a move to a position in $\mathcal{P}$.* Look at the leftmost component with odd number of 1. Change a number with 1 in that component so that there are even number of 1 in that component. You get a smaller number. Thus this is a legal move to a position in $\mathcal{P}$.

3. *Every move from a position in $\mathcal{P}$ is to a position in $\mathcal{N}$.* If $(x_1, x_2, x_3) \in \mathcal{P}$ and x_1 is changed to $x_1' < x_1$. Then the nim-sum of (x_1', x_2, x_3) cannot be zero. Otherwise, cancellation law implies that $x_1' = x_1$.

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Graph games

We give a description of combinatorial games as a game played on a directed graph.

We identify the positions in a game with vertices of the graph and moves of the game with edges of the graph.

Definition: A **directed graph** G is a pair (X, F) where X is a non-empty set of vertices (positions) and F is a function that gives for each $x \in X$ a subset of X, $F(x) \subset X$. Here $F(x)$ represents the positions to which a player may move from x (also called the followers of x). If $F(x)$ is empty, x is called a terminal position.

A two-person game may be played on such a graph $G = (X, F)$ by choosing a starting position x_0 and using the following rules

1. Player I moves first, starting at x_0.
2. Players alternate moves.
3. At position x, the player can only move to positions y where $y \in F(x)$.
4. The player who is confronted with a terminal position loses.

Remark: We assume that the graph is **progressively bounded** so that a terminal position is reached in a finite (and bounded) number of moves.

Example: subtraction game with $S = \{1, 2, 3\}$. Take $X = \{0, 1, \cdots, n\}$. $F(0) = \phi$, $F(1) = \{0\}$, $F(2) = \{0, 1\}$ and $F(k) = \{k-3, k-2, k-1\}$ $(2 \leq k \leq n)$.

The Sprague-Grundy function

For the graph (X, F), we define the Sprague-Grundy function (SG-function) g on X by

$$g(x) = \min\{n \geq 0 \,:\, n \neq g(y),\ y \in F(x)\}, \quad x \in X.$$

Note, $g(x)$ is defined recursively.

If x is a terminal position, $F(x)$ is empty and $g(x) = 0$.

For those x such that the followers are terminal positions, $g(x) = 1$.

Other values can be found inductively.

The SG-function can be used to analyze graph games.

Positions x for which $g(x) = 0$ are P-positions and all other positions are N-positions.

The winning strategy is to choose a move to a position with zero SG-function value.

Checking the 3 conditions:

1. If x is a terminal position, $g(x) = 0$.
2. At position x for which $g(x) = 0$, every follower y of x is such that $g(y) \neq 0$.
3. At positions x for which $g(x) \neq 0$, there is at least one follower y such that $g(y) = 0$.

Example: see the graph in the next page.

All terminal positions are assigned SG-value 0. There are exactly 4 terminal positions.

There is only 1 vertex all of whose followers have been assigned SG-value. This is the vertex a. Thus this vertex has SG-value 1.

Next, there are two more vertices all of whose followers have been assigned SG-value. They are vertices b and c. For vertex b, its followers have SG-value 0 and 1, so its SG-value is 2. For vertex c, its follower has SG-value 1, so its SG-value is 0.

The rest of the SG-values can be found similarly.

Figure for the example in the previous slide.

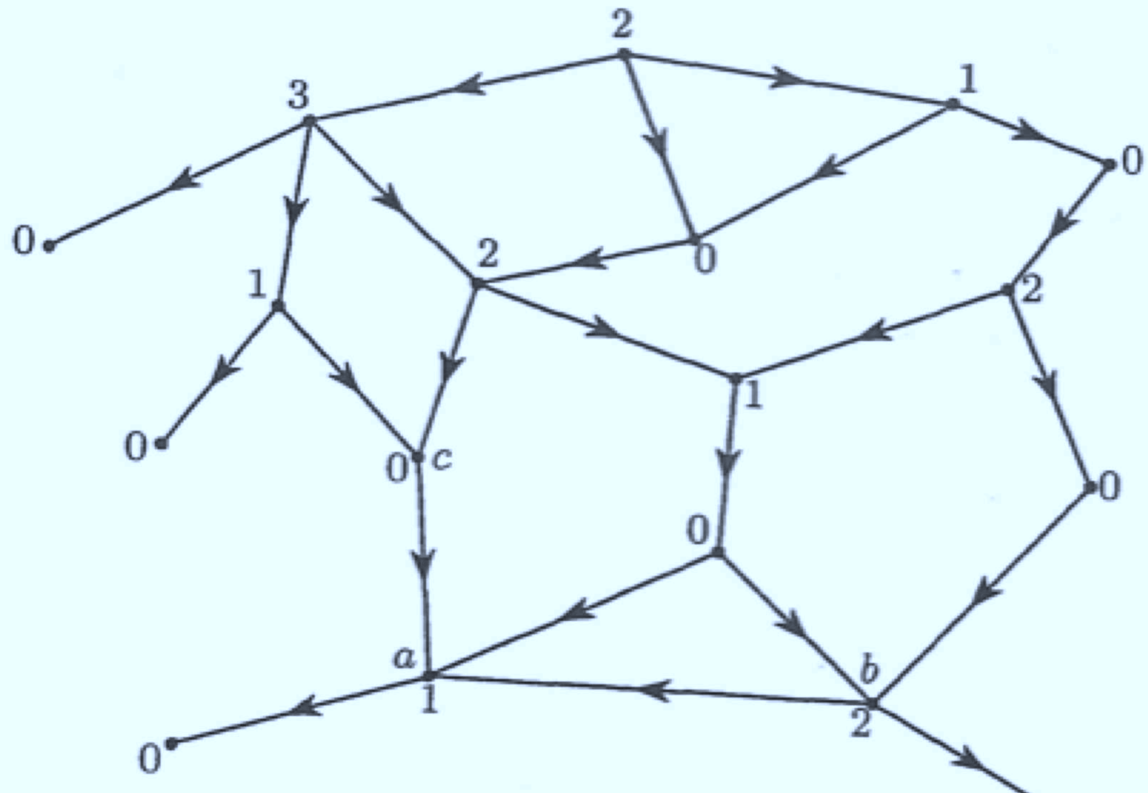

Fig. 3.2

Example: the subtraction game with subtraction set $S = \{1, 2, 3\}$

The terminal vertex 0 has SG-value 0, i.e., $g(0) = 0$.

For vertex 1, the only follower is 0 which has SG-value 0, thus $g(1) = 1$.

For vertex 2, 0 and 1 are followers. Thus $g(2) = 2$.

For vertex 3, $0, 1$ and 2 are followers. Thus $g(3) = 3$.

But for vertex 4, the followers are $1, 2, 3$ with SG-values $1, 2, 3$ respectively. Thus $g(4) = 0$.

Continuing,

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
g	0	1	2	3	0	1	2	3	0	1	2	3	0	1

Note $g(x) = x \pmod 4$.

Example: At-least half

Consider one-pile game with the rule that you must remove at least half of the counters.

The only terminal position is 0.

The SG-function is

x	0	1	2	3	4	5	6	7	8	9	10	11	12
g	0	1	2	2	3	3	3	3	4	4	4	4	4

Note $g(x) = \min\{k : 2^k > x\}$.

Sums of combinatorial games

Given several combinatorial games, one can form a new game played according to the following rules.

Players alternate moves. A move for a player consists of selecting one of the games and making a legal move in that game, leaving all other games untouched.

Play continues until all of the games have reached the terminal positions.

The player who makes the last move is the winner.

Next, we state a formal definition.

Formal definition of sum of graph games: Given n progressively bounded graph $G_1 = (X_1, F_1), \cdots, G_n = (X_n, F_n)$. One can form a new graph game, $G = (X, F)$, called the **sum** of the games, denoted by $G_1 + \cdots + G_n$.

The set X is defined by $X_1 \times \cdots \times X_n$. Thus every element $x \in X$ has the form $x = (x_1, \cdots, x_n)$ where $x_i \in X_i$.

The set of followers $F(x)$ of x is defined by

$$\begin{aligned} F(x) = F(x_1, \cdots, x_n) = & F_1(x_1) \times \{x_2\} \times \cdots \times \{x_n\} \\ & \cup \{x_1\} \times F_2(x_2) \times \cdots \times \{x_n\} \\ & \cup \cdots \\ & \cup \{x_1\} \times \cdots \times \{x_{n-1}\} \times F_n(x_n) \end{aligned}$$

Thus, a move from $(x_1, \cdots, x_n)$ consists in moving exactly one of the x_i to one of its followers $F_i(x_i)$.

Example: the 3-pile game of nim is sum of 3 one-pile nim.

The SG-function for sums of graph games

Let g_i be the SG-function for the graph game G_i, then $G = G_1 + \cdots + G_n$ has SG-function defined by

$$g(x_1, \cdots, x_n) = g_1(x_1) \oplus \cdots \oplus g_n(x_n)$$

Example: sum of subtraction games

Let $G(m)$ be the subtraction game with $S_m = \{1, \cdots, m\}$.

Note SG-function for $G(m)$ is $g_m(x) = x \pmod{m+1}$.

Consider the game $G(3) + G(5) + G(7)$ with position $(9, 10, 14)$.

How do you play?

$g(9, 10, 14) = g_3(9) \oplus g_5(10) \oplus g_7(14) = 1 \oplus 4 \oplus 6 = 3.$ (N-position)

One move is to change value of g_7 to 5. (remove 1 chip from the pile with 14 chips to 13.)

Example: Even if Not All - All if odd

Consider the one-pile game with the rule that you can remove (1) any even number of chips provided it is not the whole pile, or (2) the whole pile provided it has an odd number of chips.

There are 2 terminal positions, namely, 0 and 2. The SG-values are

x	0	1	2	3	4	5	6	7	8	9	10	11	12
g	0	1	0	2	1	3	2	4	3	5	4	6	5

Note $g(2k) = k - 1$ and $g(2k - 1) = k$ with $k \geq 1$.

Consider this game is played with three piles of sizes $10, 13$ and 20.

The SG-values are $g(10) = 4$, $g(13) = 7$ and $g(20) = 9$. Also, $4 \oplus 7 \oplus 9 = 10$. This is a N-position.

A winning move is to change the SG-value of 9 and 3. One can do this by removing 12 chips from the pile of 20.

Example: Sum of 3 different games

Game 1: Even if Not All - All if Odd with 18 chips.

Game 2: At-least Half with 17 chips.

Game 3: Game of Nim with 7 chips.

The SG-values are $8, 5, 7$ respectively. The nim-sum is 10. Thus this is a N-position.

To move to a P-position, we could change the SG-value of the first game to 2. This is the case when the pile has 3 or 6 chips. We cannot move from 18 to 3. But we can move from 18 to 6, by removing 12 chips.

Example: Take-and-Break game

A move is either (1) to remove any number from chips from one pile, or (2) to split one pile containing at least 2 chips into two non-empty piles.

Consider one pile game, we have $g(0) = 0$ and $g(1) = 1$.

Note that the followers of 2 are $0, 1, (1, 1)$, and their SG-values are $0, 1, 1 \oplus 1 = 0$. Thus, $g(2) = 2$.

The followers of 3 are $0, 1, 2, (1, 2)$, and their SG-values are $0, 1, 2, 1 \oplus 2 = 3$. Thus, $g(3) = 4$.

Continuing,

x	0	1	2	3	4	5	6	7	8	9	10	11	12
g	0	1	2	4	3	5	6	8	7	9	10	12	11

Consider the position $(2, 5, 7)$ in the above game, what is your move?

The SG-values of the components are $2, 5, 8$. And we have $2 \oplus 5 \oplus 8 = 15$. Thus this is a N-position.

We must change the SG-value 8 to 7. We can do this by splitting the pile of 7 chips into two piles with 1 and 6 chips.

Then your opponent will face the position $(1, 2, 5, 6)$, which is a P-position.

MAT 4250: Lecture 3

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Chapter 2: Two-person zero-sum games

Section 2.1: The strategic form of a game

Strategic form of a game

von Neumann, in 1928, laid the foundation for the theory of two-person zero-sum games.

A two-person zero-sum game is a game with only 2 players in which one player wins what the other loses.

The two players are called Player I and Player II.

Note that the payoff function of Player II is the negative of the payoff of Player I. So, we may restrict attention to the single payoff function of Player I, called L.

Definition: The **strategic form** of a two-person zero-sum game is given by a triplet (X, Y, A), where

1. X is a nonempty set, the set of strategies of Player I.
2. Y is a nonempty set, the set of strategies of Player II.
3. A is a real-valued function defined on $X \times Y$. (Thus, $A(x, y)$ is a real number for $x \in X$ and $y \in Y$.)

Interpretation:

Simultaneously, Player I chooses $x \in X$ and Player II chooses $y \in Y$, each unaware of the choice of the other. Then the choices are made known and I wins the amount of $A(x, y)$. If A is negative, I loses the absolute value of this amount to II.

Example: Odd or Even

Players I and II simultaneously call out one of the numbers 1 and 2. Player I wins if the sum of the numbers is odd, and Player II wins if the sum of the numbers is even. The amount paid to the winner by the loser is the sum of the two numbers.

To put this game in strategic form, we let $X = \{1, 2\}$ and $Y = \{1, 2\}$. We define A by the following table:

$$\begin{array}{ccc} & & \text{y} \\ & & \begin{array}{cc} 1 & 2 \end{array} \\ \text{x} & \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} -2 & +3 \\ +3 & -4 \end{pmatrix} \end{array}$$

It turns out that one of the players has a distinct advantage in this game. Who is this player?

Let's analyze from Player I's point of view.

Suppose Player I calls "1" 3/5-th of the time and "2" 2/5-th of the time at random. So,

- if Player II calls "1", Player I loses 2 3/5-th of the time and wins 3 2/5-th of the time. On average, he wins $-2(3/5)+3(2/5)=0$. (That is, he breaks even in the long run.)
- if Player II calls "2", Player I wins 3 3/5-th of the time and loses 4 2/5-th of the time. On average, he wins $3(3/5)-4(2/5)=1/5$.

By using this simple strategy, Player I is assured of at least break even on the average no matter what Player II does.

Can Player I fix his strategy so that he wins a positive amount no matter what II does?

Let p be the proportion of times Player I calls "1".

Let's try to choose p so that Player I wins the same amount on the average no matter what Player II calls.

If Player II calls "1", Player I's average winning is $-2p + 3(1-p)$.

If Player II calls "2", Player I's average winning is $3p - 4(1-p)$.

Setting them to equal,

$$-2p + 3(1-p) = 3p - 4(1-p) \quad \Longrightarrow \quad p = \frac{7}{12}.$$

Hence, I should call "1" with probability $7/12$ and call "2" with probability $5/12$.

In this case, I's average winning is $-2p + 3(1-p) = 1/12$.

Such a strategy that produces the same average winnings no matter what the opponent does is called an **equalizing strategy**.

Can Player I do better?

The answer is NO is Player II plays properly.

In fact, Player II can use the same strategy: call "1" with probability $7/12$ and call "2" with probability $5/12$.

Thus if Player I calls "1", II's average loss is $-2(7/12) + 3(5/12) = 1/12$. And if Player I calls "2", II's average loss is $3(7/12) - 4(5/12) = 1/12$.

Hence, I has a procedure that guarantees him at least $1/12$ on average, and II has a procedure that keeps her average loss to at most $1/12$.

$1/12$ is called the **value** of the game. This procedure is called an **optimal strategy** or **minimax strategy**.

Pure and mixed strategies

We refer to elements of X or Y as pure strategies.

The more complex entity that chooses among pure strategies at random in various proportions is called a mixed strategy.

For instance, in the example above, Player I's optimal strategy is a mixed strategy, mixing pure strategies "1" and "2" with probabilities 7/12 and 5/12 respectively.

Note that every pure strategy, $x \in X$, can be considered as the mixed strategy that chooses the pure strategy x with probability 1.

Remark: We have made an assumption that the players are only interested in their average return. Sometimes this may not be the most important interest. (We are assuming that a player is indifferent between receiving 5 million dollars outright, and receiving 10 million dollars with probability 1/2 and nothing with probability 1/2. I think everyone would prefer the 5 million.)

The Minimax Theorem

A two-person zero-sum game (X, Y, A) is said to be a **finite game** if both strategy sets X and Y are finite sets.

The following is a fundamental theorem in game theory.

The Minimax Theorem: For every finite two-person zero-sum game,

1. there is a number V, called the value of the game,
2. there is a mixed strategy for Player I such that I's average gain is at least V no matter what II does, and
3. there is a mixed strategy for Player II such that II's average loss is at most V no matter what I does.

(Remark: the game is fair if $V = 0$.)

Chapter 2: Two-person zero-sum games

Section 2.2: Matrix games

Matrix games

A finite two-person zero-sum game in strategic form (X, Y, A) is sometimes called a matrix game because the payoff function A can be represented by a matrix.

If $X = \{x_1, \cdots, x_m\}$ and $Y = \{y_1, \cdots, y_n\}$, then by the game matrix or payoff matrix we mean the matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \text{where} \quad a_{ij} = A(x_i, y_j).$$

Player I chooses a row and Player II chooses a column, and II pays I the entry in the chosen row and column.

Note that the entries of the matrix are the winnings of Player I (the row chooser) and losses of Player II (the column chooser).

A mixed strategy for Player I may be represented by an m-tuple $p = (p_1, \cdots, p_m)$ of probabilities that add to 1. If I uses the mixed strategy p and II chooses column j, then the average payoff to I is

$$\sum_{i=1}^{m} p_i a_{ij}$$

Similarly, a mixed strategy for Player II may be represented by an n-tuple $q = (q_1, \cdots, q_n)$ of probabilities that add to 1. If II uses the mixed strategy q and I chooses column i, then the average payoff to I is

$$\sum_{j=1}^{n} a_{ij} q_j$$

If I uses p and II uses q, then the average payoff to I is

$$p^T A q = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j$$

Saddle points

Now, we shall be attempting to solve games. This mean finding the value and at least one optimal strategy for each player.

Sometimes, it is easy to solve.

Saddle points: An entry a_{ij} of matrix A is a saddle point if

1. a_{ij} is the minimum of the i-th row, and
2. a_{ij} is the maximum of the j-th column.

In this case, Player I can win at least a_{ij} by choosing row i, and Player II can keep her loss to at most a_{ij} by choosing column j.

Hence a_{ij} is the value of the game.

Example: Consider the matrix game

$$A = \begin{pmatrix} 4 & 1 & -3 \\ 3 & 2 & 5 \\ 0 & 1 & 6 \end{pmatrix}$$

It is clear that the $(2,2)$-entry is a saddle point.

Thus, it is optimal for I to choose the second row and for II to choose the second column.

The value of the game is 2.

An optimal strategy for both players is $(0,1,0)$.

For large $m \times n$ matrix, it is tedious to check each entry of the matrix to see if it has the saddle point property.

It is easier to compute the minimum of each row and the maximum of each column to see if there is a match.

$$
\begin{array}{cc}
 & \begin{array}{cccc} & & & \end{array} \qquad \text{row min} \\
A = & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 2 \end{pmatrix} \qquad \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \\
\text{col max} & \begin{array}{cccc} 3 & 2 & 2 & 2 \end{array}
\end{array}
$$

No row minimum is equal to any column maximum, so there is no saddle point.

$$
\begin{array}{ccc}
 & & \text{row min} \\
B = & \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 2 \end{pmatrix} & \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \\
\text{col max} & \begin{array}{cccc} 3 & 1 & 2 & 2 \end{array} &
\end{array}
$$

In this case, the minimum of the 4-th row is equal to the maximum of the second column. So, b_{42} is a saddle point.

Solution of 2×2 matrix games

Consider the general 2×2 game matrix

$$A = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$$

To solve this game (to find the value and at least one optimal strategy for each player), we proceed as follows.

1. Test for a saddle point.
2. If there is no saddle point, solve by finding equalizing strategies.

Now, we prove the method of finding equalizing strategies of previous section works when there is no saddle point by deriving the value and the optimal strategies.

$$A = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$$

Assume there is no saddle point.

- Assume $a \geq b$. Then $b < c$ (otherwise b is a saddle point). Then $c > d$ (otherwise c is a saddle point). Then $d < a$ (otherwise d is a saddle point). Then $a > b$ (otherwise a is a saddle point). That is

$$a > b < c > d < a$$

- Assume $a \leq b$. Similarly, we have

$$a < b > c < d > a$$

Hence, if there is no saddle point, one of the above two cases hold.

$$A = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$$

Suppose I uses the mixed strategy $(p, 1-p)$. (I chooses row one with probability p.)

If II chooses column one, I's average return is $ap + d(1-p)$. If II chooses column two, I's average return is $bp + c(1-p)$.

Setting them to equal,

$$ap + d(1-p) = bp + c(1-p) \quad \Longrightarrow \quad p = \frac{c-d}{(a-b)+(c-d)}$$

If there is no saddle point, $(a-b)$ and $(c-d)$ are either positive or both negative. Hence $0 < p < 1$.

$$A = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$$

From above, I should use the strategy $(p, 1-p)$ with

$$p = \frac{c-d}{(a-b)+(c-d)}$$

So, Player I's average return is

$$v = ap + d(1-p) = \frac{ac-bd}{a-b+c-d}$$

$$A = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$$

On the other hand, suppose II uses the mixed strategy $(q, 1-q)$. (II chooses column one with probability q.)

If I chooses row one, II's average return is $aq + b(1-q)$. If I chooses row two, II's average return is $dq + c(1-q)$.

Setting them to equal,

$$aq + b(1-q) = dq + c(1-q) \quad \Longrightarrow \quad q = \frac{c-b}{(a-d)+(c-b)}$$

If there is no saddle point, $(a-d)$ and $(c-b)$ are either positive or both negative. Hence $0 < q < 1$.

$$A = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$$

From above, II should use the strategy $(q, 1-q)$ with

$$q = \frac{c-b}{(a-d)+(c-b)}$$

So, Player II's average return is

$$aq + b(1-q) = \frac{ac-bd}{a-b+c-d} = v$$

This is the same value achievable by Player I. This shows that the game has a value, and that the players have optimal strategies.

Example: $A = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$

$$p = \frac{-4-3}{-2-3-4-3} = \frac{7}{12}$$

$$q = \frac{-4-3}{-2-3-4-3} = \frac{7}{12}$$

$$v = \frac{8-9}{-2-3-4-3} = \frac{1}{12}$$

Example:

$$A = \begin{pmatrix} 0 & -10 \\ 1 & 2 \end{pmatrix}$$

$$p = \frac{2-1}{0+10+2-1} = \frac{1}{11}$$

$$q = \frac{2+10}{0+10+2-1} = \frac{12}{11}$$

But q should be between 0 and 1. What happened? There is a saddle point a_{21}.

Removing dominated strategies

Sometimes, large matrix game may be reduced in size by deleting rows and columns that are obviously bad for the player who uses them.

<u>Definition</u>: We say the i-th row of a matrix A dominates the k-th row if $a_{ij} \geq a_{kj}$ for all j. We say the i-th row of a matrix A strictly dominates the k-th row if $a_{ij} > a_{kj}$ for all j.

<u>Definition</u>: Similarly, we say the j-th column of a matrix A dominates (strictly dominates) the k-th column if $a_{ij} \leq a_{ik}$ (resp. $a_{ij} < a_{ik}$) for all i.

Anything I can achieve using a dominated row can be achieved at least as well using the row that dominates it. Thus, dominated rows may be deleted from the matrix.

Similarly, dominated columns may be removed.

Thus, removal of a dominated row or column does not change the value of a game.

But there may exist an optimal strategy that uses a dominated row or column. (see Assignment 2.) If so, removal of that row or column will also remove the use of that optimal strategy.

In case of removal of a strictly dominated row or column, the set of optimal strategies does not change.

We can iterate the above procedure and successively remove several rows and columns. Consider

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{pmatrix} \Rightarrow A_1 = \begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 4 & 1 \end{pmatrix} \Rightarrow A_2 = \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$$

Note, the last column is dominated by the middle column. Removing the last column, we get A_1. Now, first row is dominated by the last row, so removing first row, we get A_2.

Thus, we obtain a 2×2 game with no saddle point. Solving

$$p = \frac{3}{4}, \quad q = \frac{1}{4}, \quad v = \frac{7}{4}$$

Hence, I's optimal strategy in the original game is $(0, 3/4, 1/4)$ and II's is $(1/4, 3/4, 0)$.

A row (column) may also be removed if it is dominated by a probability combination of other rows (columns).

If for some $0 < p < 1$,

$$pa_{i_1 j} + (1-p)a_{i_2 j} \geq a_{kj}, \quad \forall j$$

then the k-th row is dominated by the mixed strategy that chooses row i_1 with probability p and row i_2 with probability $1-p$.

Player I can do at least as well using this mixed strategy instead of choosing row k.

Similarly argument can be used for columns.

Example:

$$A = \begin{pmatrix} 0 & 4 & 6 \\ 5 & 7 & 4 \\ 9 & 6 & 3 \end{pmatrix} \Rightarrow A_1 = \begin{pmatrix} 0 & 6 \\ 5 & 4 \\ 9 & 3 \end{pmatrix} \Rightarrow A_2 = \begin{pmatrix} 0 & 6 \\ 9 & 3 \end{pmatrix}$$

The middle column is dominated by the first and the third columns taken with probability 1/2 each. Removing the central column, we get A_1. Then the middle row of A_1 is dominated by the combination of top row with probability 1/3 and bottom row with probability 2/3. Removing middle row, we get A_2.

Solving, we get $V = 9/2$.

Solving $2 \times n$ and $m \times 2$ games

These games can be solved with the aid of a graphical representation. For example, consider

$$\begin{array}{cc} p & \\ 1-p & \end{array} \begin{pmatrix} 2 & 3 & 1 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix}$$

I chooses row 1 with prob. p and row 2 with prob. $1-p$.

The average payoffs for I are $2p + 4(1-p)$, $3p + (1-p)$, $p + 6(1-p)$ and $5p$ when II chooses column $1, 2, 3$ and 4 resp.

For fixed p, I can be sure that his average winnings is at least the minimum of these 4 functions evaluated at p, that is,

$$\min\{2p + 4(1-p),\ 3p + (1-p),\ p + 6(1-p),\ 5p\}$$

this is called the lower envelope of these functions.

Since I wants to maximize his guaranteed average winnings, he wants to find p that achieves the maximum of this lower envelope.

See the figure next page.

This max occurs at the intersection of the lines for columns 2 and 3.

Thus, this essentially involves solving the game in which II is restricted to columns 2 and 3. That is, the game

$$\begin{pmatrix} 3 & 1 \\ 1 & 6 \end{pmatrix}$$

The value of this game is $v = 17/7$ and I's optimal strategy is $(5/7, 2/7)$, and II's optimal strategy is $(5/7, 2/7)$.

Hence, in the original game, I's optimal strategy is $(5/7, 2/7)$, and II's optimal strategy is $(0, 5/7, 2/7, 0)$. The value is $17/7$.

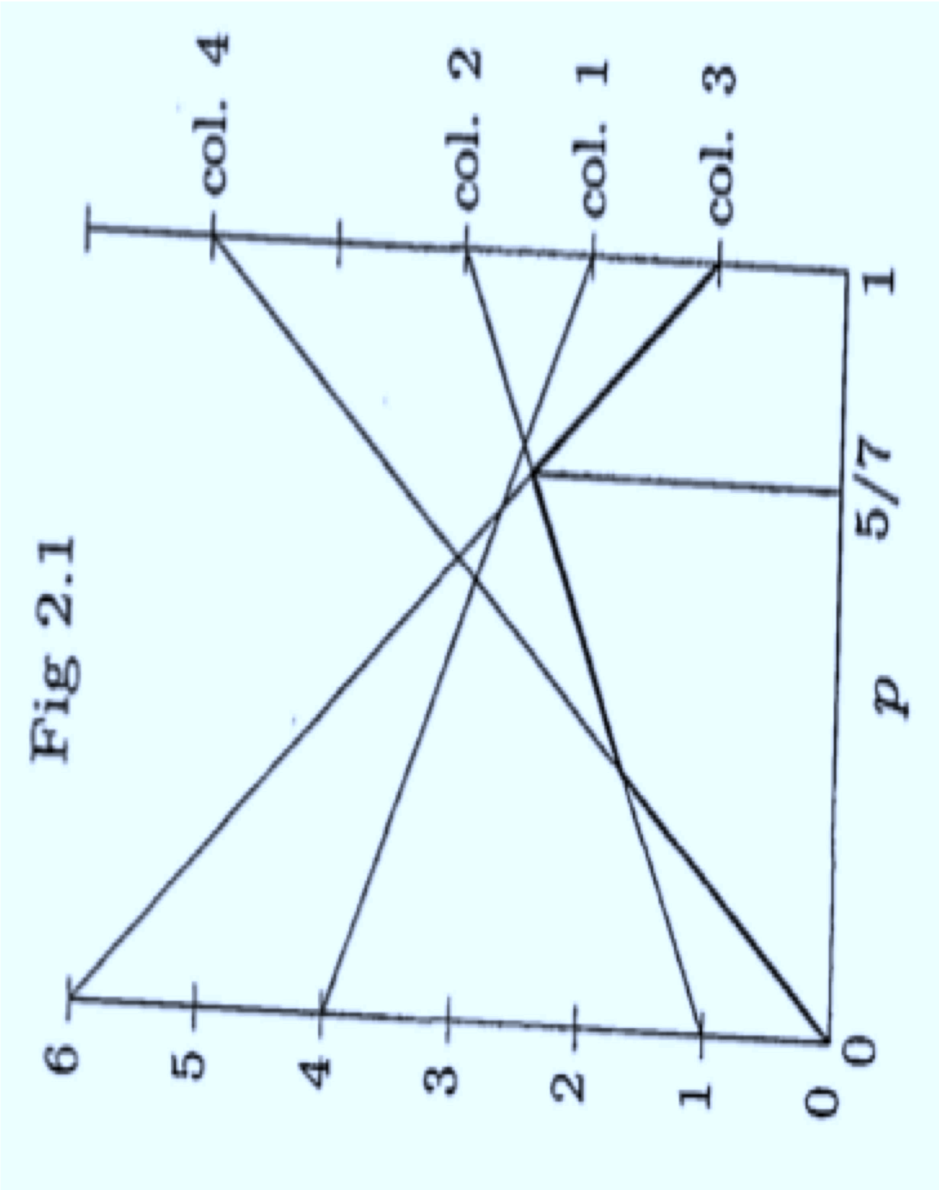


Fig 2.1

Remark: referring to the figure in previous page.

The line for column 1 plays no role in the lower envelope. This is actually a test for domination. Column 1 is dominated by columns 2 and 3 taken with probability 1/2 each.

The line for column 4 does appear in the lower envelope, and column 4 cannot be dominated.

Example: $m \times 2$ game, refer to the figure next page

II chooses column 1 with prob. q and column 2 with prob. $1 - q$.

The average loss for II are $q + 5(1 - q)$, $4q + 4(1 - q)$ and $6q + 2(1 - q)$ when I chooses row $1, 2$ and 3 resp.

For fixed q, II can be sure that his average loss is at most the maximum of these 3 functions evaluated at q, that is,

$$\max\{q + 5(1 - q),\ 4q + 4(1 - q),\ 6q + 2(1 - q)\}$$

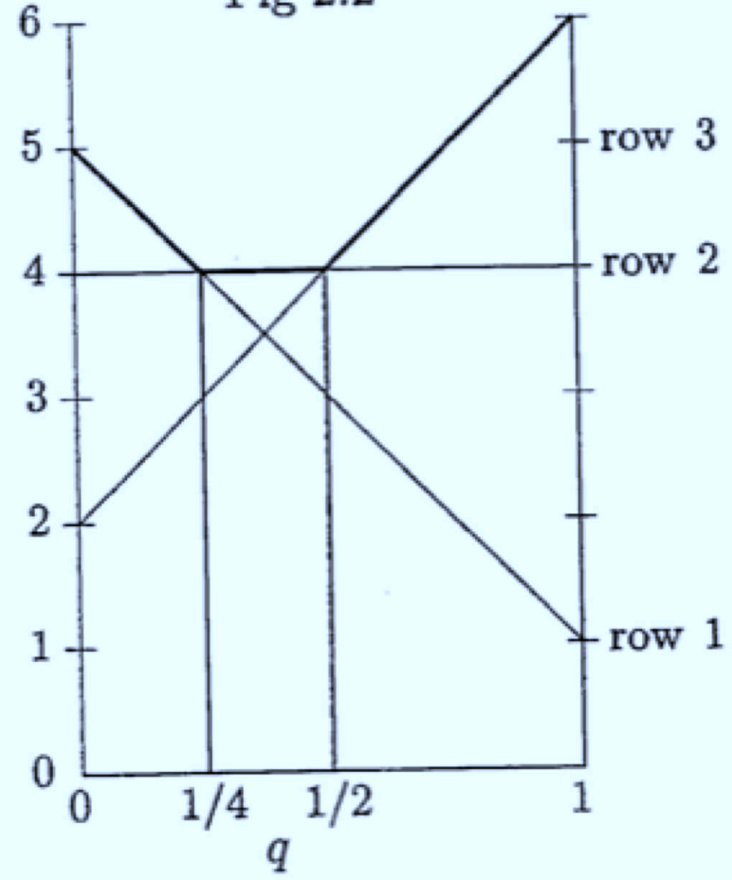
this is called the upper envelope of these functions.

II's wants to minimize this maximum loss.

From graph, II can take q between $1/4$ and $1/2$. The value of the game is 4. And I has an optimal strategy $(0, 1, 0)$.

$$\begin{array}{cc} q & 1-q \\ \end{array}$$
$$\begin{pmatrix} 1 & 5 \\ 4 & 4 \\ 6 & 2 \end{pmatrix}$$

Fig 2.2



MAT 4250: Lecture 4

Eric Chung

Chapter 2: Two-person zero-sum games

Section 2.3: The Principle of Indifference

Consider a matrix game with $m \times n$ matrix A.

If I uses the mixed strategy $p = (p_1, \cdots, p_m)$ and II uses column j, then I's average payoff is $\sum_{i=1}^{m} p_i a_{ij}$.

If V is the value of the game, an optimal strategy p for I is characterized by the property that I's payoff is at least V no matter what column II uses, i.e.,

$$\sum_{i=1}^{m} p_i a_{ij} \geq V, \qquad j = 1, 2, \cdots, n$$

Similarly, a strategy $q = (q_1, \cdots, q_n)$ is optimal for II iff

$$\sum_{j=1}^{n} a_{ij} q_j \leq V, \qquad i = 1, 2, \cdots, m$$

Assume that both players use their optimal strategies.

Note that the average payoff for both players is
$\sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j = p^T A q$.

By above,

$$V = \sum_{j=1}^{n} V q_j \leq \sum_{j=1}^{n} \Big\{ \sum_{i=1}^{m} p_i a_{ij} \Big\} q_j = \sum_{i=1}^{m} p_i \Big\{ \sum_{j=1}^{n} a_{ij} q_j \Big\} \leq \sum_{i=1}^{m} p_i V = V$$

Hence, the average payoff for both players is V.

Question: if II uses the optimal strategy q, can you find a strategy p that achieve the value V? (recall $\sum_{j=1}^{n} a_{ij} q_j \leq V, \quad i = 1, 2, \cdots, m$)

Question: if I uses the optimal strategy p, can you find a strategy q that achieve the value V? (recall $\sum_{i=1}^{m} p_i a_{ij} \geq V, \quad j = 1, 2, \cdots, n$)

Equilibrium Theorem

Theorem: Consider a matrix game with $m \times n$ matrix A. Let p and q be optimal strategies for I and II resp. Then

$$\sum_{j=1}^{n} a_{ij} q_j = V, \quad \text{for all } i \text{ with } p_i > 0$$

and

$$\sum_{i=1}^{m} p_i a_{ij} = V, \quad \text{for all } j \text{ with } q_j > 0$$

Proof. Suppose there is k such that $p_k > 0$ and $\sum_{j=1}^{n} a_{kj} q_j \neq V$. Then $\sum_{j=1}^{n} a_{kj} q_j < V$. By above

$$V = \sum_{i=1}^{m} p_i \Big\{ \sum_{j=1}^{n} a_{ij} q_j \Big\} < \sum_{i=1}^{m} p_i V = V$$

which is a contradiction.

Remarks:

1. Another way of saying the first conclusion: if there exists an optimal strategy p for I with positive probability to row i, then every optimal strategy of II gives I the value of the game if I chooses row i.

2. The theorem suggests that I should try to find a solution p to those equations $\sum_{i=1}^{m} p_i a_{ij} = V$ with $q_j > 0$. In this case, I has a strategy what makes II indifferent as to which of the pure strategies to use.

3. Similar argument works for II. This is called the **Principle of Indifference**.

Example: Consider the Odd-or-Even game in which both players call out the numbers $0, 1, 2$.

The matrix is

$$\begin{array}{cc} & \text{Even} \\ \text{Odd} & \begin{pmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{pmatrix} \end{array}$$

Assume II's optimal strategy gives +ve weights to each column.

I's optimal strategy p satisfies

$$p_2 - 2p_3 = V, \;\; p_1 - 2p_2 + 3p_3 = V, \;\; -2p_1 + 3p_2 - 4p_3 = V$$

Note V is unknown. Need one more equation, $p_1 + p_2 + p_3 = 1$.

Solving the equations, we get $p = (1/4, 1/2, 1/4)$ and $V = 0$.

From above, we see that the value of the game is at least 0, if our assumption is correct.

Similarly, if we assume I's optimal strategy gives +ve weights to each row. Then II's optimal strategy q satisfies

$$q_2 - 2q_3 = V, \;\; q_1 - 2q_2 + 3q_3 = V, \;\; -2q_1 + 3q_2 - 4q_3 = V$$

Solving, we get $q = (1/4, 1/2, 1/4)$ and $V = 0$.

Hence, II has a strategy q that keeps his average loss to zero no matter what I does.

Thus, the value of the game is zero and the above p and q are optimal strategies for I and II. This game is fair.

Nonsingular game matrices

Let A be a $m \times m$ nonsingular matrix.

Assume that I has optimal strategy giving +ve weight to each row.

By principle of indifference, II's optimal strategy q satisfies

$$\sum_{j=1}^{m} a_{ij} q_j = V, \quad i = 1, 2, \cdots, m.$$

Notation: $\mathbf{1} = (1, 1, \cdots, 1)^T$.

Then we have $Aq = V\mathbf{1}$. Thus $V \neq 0$ since A is nonsingular.

And we have $q = VA^{-1}\mathbf{1}$.

To find V, use $\sum_{j=1}^{m} q_j = 1$ or equivalently $\mathbf{1}^T q = 1$.

We have $1 = \mathbf{1}^T q = V\mathbf{1}^T A^{-1}\mathbf{1} \quad \Longrightarrow \quad V = 1/\mathbf{1}^T A^{-1}\mathbf{1}$

Hence $q = A^{-1}\mathbf{1}/\mathbf{1}^T A^{-1}\mathbf{1}$.

Note: if some components of q is -ve, our assumption is wrong.

Suppose $q_j \geq 0$ for all j.

Now, we could use the same reasoning to find an optimal strategy p for I, and the result is the same, namely, $p = A^{-T}\mathbf{1}/\mathbf{1}^T A^{-1}\mathbf{1}$.

If p is non-negative, then both p and q are optimal strategies, that guarantee both players the average payoff V.

We summarize the result in the theorem.

Theorem: Assume the $m \times m$ matrix A is non-singular and $\mathbf{1}^T A^{-1}\mathbf{1} \neq 0$. Then the game with matrix A has value $V = 1/\mathbf{1}^T A^{-1}\mathbf{1}$ and optimal strategies $p = VA^{-T}\mathbf{1}$ and $q = VA^{-1}\mathbf{1}$ provided $p \geq 0$ and $q \geq 0$.

Note: if the value of a game is zero, the above method cannot be applied, because $Aq = V\mathbf{1}$ implies that A is singular.

Add a +ve constant to all entries to make the game value +ve.

$$\begin{pmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{pmatrix} \text{ adding one becomes } A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 4 \\ -1 & 4 & -3 \end{pmatrix}$$

Then we have $A^{-1} = \dfrac{1}{16}\begin{pmatrix} 13 & -2 & -7 \\ -2 & 4 & 6 \\ -7 & 6 & 5 \end{pmatrix}$

So, $\mathbf{1}^T A^{-1} \mathbf{1} = 1$.

Hence $V = 1$ and $p = (1/4, 1/2, 1/4)$ and $q = (1/4, 1/2, 1/4)$.

Diagonal games

Consider matrix game with game matrix A square and diagonal

$$A = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_m \end{pmatrix}$$

Assume that all diagonal entries $d_i > 0$.

Note $V = 1/\mathbf{1}^T A^{-1}\mathbf{1} = (\sum_{i=1}^m 1/d_i)^{-1}$.

And $p = VA^{-T}\mathbf{1} = V(1/d_1, \cdots, 1/d_m)^T$.

Similarly, $q = VA^{-1}\mathbf{1} = V(1/d_1, \cdots, 1/d_m)^T$.

Since $p > 0$ and $q > 0$, p and q are optimal strategies and V is the value of the game.

Example: consider the diagonal game matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

We have $V = (1 + 1/2 + 1/3 + 1/4)^{-1} = 12/25.$

And $p = 12/25(1, 1/2, 1/3, 1/4) = (12/25, 6/25, 4/25, 3/25).$

Similarly, $q = (12/25, 6/25, 4/25, 3/25).$

Triangular games

Consider the triangular game matrix $T = \begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Following the above discussion, assume II has optimal strategy with positive weight in each entry.

Then optimal strategy p for I satisfies $\sum_{i=1}^{m} p_i a_{ij} = V$, thus

$p_1 = V, -2p_1 + p_2 = V, 3p_1 - 2p_2 + p_3 = V, -4p_1 + 3p_2 - 2p_3 + p_4 = V$

Solving $p_1 = V,\ p_2 = 3V,\ p_3 = 4V,\ p_4 = 4V$

Since $\sum_{i=1}^{m} p_i = 1$, we get $V = 1/12$. And $p = (1/12, 1/4, 1/3, 1/3)$.

Similar argument shows that $q = (1/3, 1/3, 1/4, 1/12)$.

Symmetric games

A game is symmetric if the rules don't distinguish the players.

For symmetric games, both players have the same options (hence the game matrix is square).

The payoff for I choosing i-th row and II choosing j-column is the negative of the payoff for I choosing j-th row and II choosing i-column, thus, $a_{ij} = -a_{ji}$.

This means that the game matrix A is skew-symmetric, $A = -A^T$.

Definition: A finite game is said to be **symmetric** if its game matrix is square and skew-symmetric.

Note: A game is symmetric if after some rearrangement of the rows and columns the game matrix is skew-symmetric.

Example: paper-scissors-rock

Both players simultaneously display one of the 3 objects: paper, scissors or rock

If the two players choose the same object, there is no payoff.

If they choose different objects, then scissors win over paper, rock wins over scissors and paper wins over rock.

The game matrix is

$$
\begin{array}{cc}
 & \begin{array}{ccc} \textit{paper} & \textit{scissors} & \textit{rock} \end{array} \\
\begin{array}{c} \textit{paper} \\ \textit{scissors} \\ \textit{rock} \end{array} & \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}
\end{array}
$$

The matrix is skew-symmetric, thus the game is symmetric.

Another example: matching pennies

Two players simultaneously choose to show a penny with either the heads or the tails side facing up.

Player I wins if the choices match, otherwise Player II wins.

The game matrix is

$$\begin{array}{cc} & \begin{array}{cc} \quad heads & tails \end{array} \\ \begin{array}{c} heads \\ tails \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}$$

Even though there is a great deal of symmetry, we do not call this a symmetric game. (as the matrix is not skew-symmetric)

We expect a symmetric game to be fair. That is the value $V = 0$.

Theorem: A finite symmetric game has value zero. Any strategy optimal for one player is also optimal for the other.

Proof: Let p be an optimal strategy for I. Suppose II uses the same strategy. Then the payoff is $p^T Ap$. But
$(p^T Ap)^T = p^T A^T p = -p^T Ap$. Thus, $p^T Ap = 0$.

This shows that $V \leq 0$.

A symmetric argument shows that $V \geq 0$. Hence $V = 0$.

Suppose p is optimal for I. Then $\sum_{i=1}^{m} p_i a_{ij} \geq 0$ for all j.

Then $\sum_{j=1}^{m} a_{ij} p_j = -\sum_{j=1}^{m} p_j a_{ji} \leq 0$. Hence p is optimal for II.

The other case can be done similarly.

Example: Mendelsohn games.

Both players simultaneously choose an integer. They want to choose an integer larger but not too much larger than the opponent.

For example, they choose integer between 1 and 100. If the numbers are equal, no payoff. The player who chooses a number one larger than the opponent wins 1. The payer who chooses a number two or more larger than the opponent loses 2.

What is the game matrix ?

Here is the game matrix

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \cdots \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{array} &
\begin{pmatrix}
0 & -1 & 2 & 2 & 2 & \cdots \\
1 & 0 & -1 & 2 & 2 & \cdots \\
-2 & 1 & 0 & -1 & 2 & \cdots \\
-2 & -2 & 1 & 0 & -1 & \cdots \\
-2 & -2 & -2 & 1 & 0 & \cdots \\
\vdots & & & & & \ddots
\end{pmatrix}
\end{array}
$$

This game is symmetric, so the value is zero and players have identical optimal strategies.

Note that row 1 dominates rows $4, 5, 6, \cdots$.

We only need to consider the upper left 3×3 submatrix.

Consider the upper left 3×3 submatrix.

$$\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

Assume that I has optimal strategy p so that $p_1 > 0, p_2 > 0, p_3 > 0$. (so, q has the same condition.)

By the principle of indifference, we have

$$p_2 - 2p_3 = 0,\ -p_1 + p_3 = 0,\ 2p_1 - p_2 = 0$$

Together with the condition $p_1 + p_2 + p_3 = 1$, we have

$$p_1 = 1/4,\ p_2 = 1/2,\ p_3 = 1/4$$

Hence, the optimal strategies are

$$p = q = (1/4, 1/2, 1/4, 0, 0, \cdots)$$

Invariance

Consider the game of matching pennies. Two players simultaneously choose heads or tails. Player I wins if the choices match and Player II wins otherwise.

There does not seem to be much of a reason for either players to choose heads instead of tails. In fact, the problem is the same if the names of heads and tails are interchanged.

In other words, the problem is **invariant** under interchanging the names of pure strategies.

We will define the notion of invariance, and show that in the search of minimax strategy, a player may restrict attention to invariant strategies.

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We look at the problem from Player II's viewpoint.

Let Y be the pure strategy set (finite) for Player II.

A transformation $g : Y \to Y$ is said to be *onto* if for every $y_1 \in Y$, there is $y_2 \in Y$ such that $g(y_2) = y_1$

A transformation $g : Y \to Y$ is said to be *one-to-one* if $g(y_1) = g(y_2)$ implies $y_1 = y_2$.

We assume all duplicate pure strategies have been removed, namely,

$$A(x', y) = A(x'', y) \quad \forall y \in Y \quad \Longrightarrow \quad x' = x''$$

$$A(x, y') = A(x, y'') \quad \forall x \in X \quad \Longrightarrow \quad y' = y''$$

Definition: Let $G = (X, Y, A)$ be a finite game, and let $g : Y \to Y$ be a one-to-one and onto transformation. The game G is said to be **invariant** under g if for every $x \in X$ there is a <u>unique</u> $x' \in X$ s.t.

$$A(x, y) = A(x', g(y)), \quad \forall y \in Y$$

Recall, from the above definition,

$$A(x,y) = A(x', g(y)), \quad \forall y \in Y$$

Observe that x' depends on g and x only. We write $x' = \overline{g}(x)$.
Thus,

$$A(x,y) = A(\overline{g}(x), g(y)), \quad \forall y \in Y$$

Note that $\overline{g}$ is a one-to-one transformation, since if $\overline{g}(x_1) = \overline{g}(x_2)$,

$$A(x_1, y) = A(\overline{g}(x_1), g(y)) = A(\overline{g}(x_2), g(y)) = A(x_2, y)$$

for all $y \in Y$. Hence $x_1 = x_2$.

Since X is finite, $\overline{g}$ is also onto.

Lemma: Let $G = (X, Y, A)$ be a finite game. If G is invariant under g, then G is also invariant under g^{-1}.

Proof. Note $A(x, y) = A(\overline{g}(x), g(y))$ for all $x \in X$ and $y \in Y$. Taking $x = \overline{g}^{-1}(x)$ and $y = g^{-1}(y)$, we have $A(\overline{g}^{-1}(x), g^{-1}(y)) = A(x, y)$. This implies G is invariant under g^{-1}. Moreover, $\overline{g^{-1}} = \overline{g}^{-1}$.

Lemma: Let $G = (X, Y, A)$ be a finite game. If G is invariant under g_1 and g_2, then G is invariant under the composition $g_2 g_1$.

Proof. Since G is invariant under g_2, $A(x, y) = A(\overline{g_2}(x), g_2(y))$ for all $x \in X$ and $y \in Y$. Taking $x = \overline{g_1}(x)$ and $y = g_1(y)$,

$$A(x, y) = A(\overline{g_2}(\overline{g_1}(x)), g_2(g_1(y))) = A(\overline{g_2}\,\overline{g_1}(x), g_2 g_1(y)), \ \forall x, y$$

So, G is invariant under $g_2 g_1$. Moreover, $\overline{g_2 g_1} = \overline{g_2}\,\overline{g_1}$.

Recall, if G is invariant under g, G is also invariant under g^{-1}, and if G is invariant under g_1 and g_2, G is also invariant under $g_2 g_1$.

Hence, the class of transformations g on Y, under which the problem is invariant, forms a group $\mathcal{G}$. (Composition is the multiplication operator and the identity element is the identity transformation $e(y) = y$.)

Similarly, the set $\overline{\mathcal{G}}$ of the corresponding transformations $\overline{g}$ is also a group. (Composition is the multiplication operator and the identity element is the identity transformation $\overline{e}(x) = x$.)

From the above two lemmas, we have $\overline{g^{-1}} = \overline{g}^{-1}$ and $\overline{g_2 g_1} = \overline{g_2}\,\overline{g_1}$. Thus, the groups $\mathcal{G}$ and $\overline{\mathcal{G}}$ are isomorphic. They are indistinguishable.

Definition: A finite game $G = (X, Y, A)$ is said to be invariant under a group $\mathcal{G}$ if for each $g \in \mathcal{G}$,

$$A(x, y) = A(\overline{g}(x), g(y)), \quad \forall x \in X, y \in Y$$

We now define what it mean for a mixed strategy q for II is invariant under a group $\mathcal{G}$.

Definition: Given a finite game $G = (X, Y, A)$ that is invariant under the group $\mathcal{G}$. A mixed strategy $q = (q(1), \cdots, q(n))$ for II is said to invariant under $\mathcal{G}$ if

$$q(g(y)) = q(y), \quad \forall y \in Y, \ g \in \mathcal{G}$$

Similarly, a mixed strategy $p = (p(1), \cdots, p(m))$ for I is said to invariant under $\overline{\mathcal{G}}$ if

$$p(\overline{g}(x)) = p(x), \quad \forall x \in X, \ \overline{g} \in \overline{\mathcal{G}}$$

Two points y_1 and y_2 are said to be equivalent if there exists $g \in \mathcal{G}$ such that $y_2 = g(y_1)$.

Note this is an equivalence relation. The set $E_y = \{y' \,:\, g(y') = y \text{ for some } g \in \mathcal{G}\}$ is called an equivalence class, or an orbit.

Thus y_1 and y_2 are equivalent if they lie on the same orbit.

Hence a mixed strategy q is invariant if it is constant of orbits.

Now we state and prove a main theorem.

Theorem: If a finite game $G = (X, Y, A)$ is invariant under a group $\mathcal{G}$, then there exists invariant optimal strategies for the players.

Proof. We show that II has an invariant optimal strategy. Since the game is finite, there is a value V and an optimal strategy q^* for II.

$$\sum_{y \in Y} A(x, y) q^*(y) \leq V, \quad \forall x \in X$$

We will show there is an invariant strategy $\tilde{q}$ satisfying the same condition. Let $N = |\mathcal{G}|$ be the number of elements in $\mathcal{G}$. Define

$$\tilde{q}(y) = \frac{1}{N} \sum_{g \in \mathcal{G}} q^*(g(y))$$

Then $\tilde{q}$ is invariant since for each $g' \in \mathcal{G}$

$$\tilde{q}(g'(y)) = \frac{1}{N} \sum_{g \in \mathcal{G}} q^*(g(g'(y))) = \frac{1}{N} \sum_{g \in \mathcal{G}} q^*(g(y)) = \tilde{q}(y)$$

Moreover,

$$\begin{aligned}
\sum_{y\in Y} A(x,y)\tilde{q}(y) &= \sum_{y\in Y} A(x,y)\frac{1}{N}\sum_{g\in\mathcal{G}} q^*(g(y)) \\
&= \frac{1}{N}\sum_{g\in\mathcal{G}}\sum_{y\in Y} A(x,y)q^*(g(y)) \\
&= \frac{1}{N}\sum_{g\in\mathcal{G}}\sum_{y\in Y} A(\overline{g}(x),g(y))q^*(g(y)) \\
&= \frac{1}{N}\sum_{g\in\mathcal{G}}\sum_{y\in Y} A(\overline{g}(x),y)q^*(y) \\
&\leq \frac{1}{N}\sum_{g\in\mathcal{G}} V \\
&= V
\end{aligned}$$

Example: consider matching pennies $G = (X, Y, A)$

$X = Y = \{1, 2\}$, $A(1, 1) = A(2, 2) = 1$, $A(1, 2) = A(2, 1) = -1$

Let $\mathcal{G} = \{e, g\}$ be a group where $g(1) = 2$, $g(2) = 1$. Note that G is invariant under this group.

The mixed strategy $q = (q(1), q(2))$ is invariant if $g(1) = g(2)$

But $g(1) + g(2) = 1$, we have $q(1) = q(2) = 1/2$. This is the only invariant mixed strategy for II, hence it is an optimal strategy.

Example: paper(1)-scissors(2)-rock(3)

$X = Y = \{1, 2, 3\}$, $A(1, 1) = A(2, 2) = A(3, 3) = 0$

$A(1, 2) = A(2, 3) = A(3, 1) = -1$, $A(2, 1) = A(3, 2) = A(1, 3) = 1$

The game is invariant under the group $\mathcal{G} = \{e, g, g^2\}$ where $g(1) = 2, g(2) = 3, g(3) = 1$

The mixed strategy $q = (q(1), q(2), q(3)$ is invariant if $q(1) = q(2)$ and $q(2) = q(3)$. Hence $q = (1/3, 1/3, 1/3)$. This is the only invariant strategy, thus it is an optimal strategy.

Example: a simple military game

- Two countries, I and II, aim at capturing two posts
- I has 4 units, and II has 3 units
- The country sending the most units to either post captures the post, and all units sent by the other country
- The country will get 1 point for the post, and 1 point for each captured unit.
- There is no payoff if both countries send the same number of unit to a post.

I has 5 pure strategy, $X = \{(4,0),(3,1),(2,2),(1,3),(0,4)\}$

II has 4 pure strategy, $Y = \{(3,0),(2,1),(1,2),(0,3)\}$

The payoff matrix is

$$
\begin{array}{cc}
 & \begin{array}{cccc} (3,0) & (2,1) & (1,2) & (0,3) \end{array} \\
\begin{array}{c} (4,0) \\ (3,1) \\ (2,2) \\ (1,3) \\ (0,4) \end{array} &
\begin{pmatrix}
4 & 2 & 1 & 0 \\
1 & 3 & 0 & -1 \\
-2 & 2 & 2 & -2 \\
-1 & 0 & 3 & 1 \\
0 & 1 & 2 & 4
\end{pmatrix}
\end{array}
$$

This is hard to solve in general. Note it cannot be solve by removing dominated strategies.

It can be solved by invariance. It involves the symmetry of the two posts.

We define the group $\mathcal{G} = \{e, g\}$ where

$$g((3,0)) = (0,3),\ g((0,3)) = (3,0),\ g((2,1)) = (1,2),\ g((1,2)) = (2,1)$$

and the corresponding group $\overline{\mathcal{G}} = \{\overline{e}, \overline{g}\}$ where

$$\overline{g}((4,0)) = (0,4),\ \overline{g}((0,4)) = (4,0),\ \overline{g}((3,1)) = (1,3),\ \overline{g}((1,3)) = (3,1)$$

$$\overline{g}((2,2)) = (2,2)$$

Note that the orbits for II are $\{(3,0),(0,3)\}$ and $\{(2,1),(1,2)\}$

A strategy q for II is invariant if $q((3,0)) = q((0,3))$ and $q((2,1)) = q((1,2))$

Similarly, a strategy p for I is invariant if $p((4,0)) = p((0,4))$ and $p((3,1)) = p((1,3))$

We reduce II's strategy space to two elements

- $(3,0)^*$: use $(3,0)$ and $(0,3)$ with probability $1/2$ each
- $(2,1)^*$: use $(2,1)$ and $(1,2)$ with probability $1/2$ each

We reduce I's strategy space to three elements

- $(4,0)^*$: use $(4,0)$ and $(0,4)$ with probability $1/2$ each
- $(3,1)^*$: use $(3,1)$ and $(1,3)$ with probability $1/2$ each
- $(2,2)$: use $(2,2)$

The new payoff matrix is

$$\begin{array}{c} \\ (4,0)^* \\ (3,1)^* \\ (2,2) \end{array} \begin{array}{c} \begin{array}{cc} (3,0)^* & (2,1)^* \end{array} \\ \begin{pmatrix} 2 & 1.5 \\ 0 & 1.5 \\ -2 & 2 \end{pmatrix} \end{array}$$

(To compute the upper left entry, note that the 4 corner entries of the original matrix appear with probability 1/4 each.)

To solve this game, we see that the middle row is dominated by the top row. The matrix becomes

$$\begin{array}{c} \\ (4,0)^* \\ (2,2) \end{array} \begin{array}{c} \begin{array}{cc} (3,0)^* & (2,1)^* \end{array} \\ \begin{pmatrix} 2 & 1.5 \\ -2 & 2 \end{pmatrix} \end{array}$$

Solving this 2×2 game, we get

$$p = (8/9, 1/9) \quad q = (1/9, 8/9) \quad V = 14/9$$

Hence the optimal strategies for the original game are

$$p = (4/9, 0, 1/9, 0, 4/9) \quad q = (1/18, 4/9, 4/9, 1/18)$$

MAT 4250: Lecture 5

Eric Chung

Chapter 2: Two-person zero-sum games

Section 2.4: Solving finite games

Best responses

Consider the game (X, Y, A) where A is $m \times n$ matrix.

X and Y are sets of pure strategies.

Define sets of mixed strategies as follows.

$$X^* = \{p = (p_1, \cdots, p_m)^T \ : \ p_i \geq 0, \ \sum_{i=1}^{m} p_i = 1\}$$

$$Y^* = \{q = (q_1, \cdots, q_n)^T \ : \ q_j \geq 0, \ \sum_{j=1}^{n} q_j = 1\}$$

The unit vector $e_k \in X^*$ is regarded as pure strategy of choosing row k, and similarly, the unit vector $e_k \in Y^*$ is regarded as pure strategy of choosing column k.

Hence we say $X \subset X^*$ and $Y \subset Y^*$.

Suppose it is known that II is going to use a particular $q \in Y^*$.

Then I would choose row i that maximize

$$\sum_{j=1}^{n} a_{ij} q_j = (Aq)_i$$

This is the same as choosing $p \in X^*$ that maximize $p^T Aq$. His average payoff is

$$\max_{1 \le i \le m} \sum_{j=1}^{n} a_{ij} q_j = \max_{p \in X^*} p^T Aq$$

(Since $X \subset X^*$, $\max_{1 \le i \le m} \sum_{j=1}^{n} a_{ij} q_j \le \max_{p \in X^*} p^T Aq$. On the other hand, $p^T Aq \le \sum_{i=1}^{m} p_i (Aq)_i \le \max_{1 \le i \le m} \sum_{j=1}^{n} a_{ij} q_j$.)

Any $p \in X^*$ that achieves the above maximum is called a **best response** or a **Bayes strategy** against q.

There exists pure Bayes strategy against q.

Similarly, if it is known that I is going to use a particular $p \in X^*$.

Then II would choose column j that minimize

$$\sum_{i=1}^{m} p_i a_{ij} = (p^T A)_j$$

or $q \in Y^*$ that minimize $p^T A q$. His average payoff is

$$\min_{1 \le j \le n} \sum_{i=1}^{m} p_i a_{ij} = \min_{q \in Y^*} p^T A q$$

Any $q \in Y^*$ that achieves the above minimum is called a **best response** or a **Bayes strategy** against p.

Upper and lower value

Suppose that II is required to announce his choice of $q \in Y^*$.

Then I would use his Bayes strategy against q and II would lose the following amount

$$\max_{1 \leq i \leq m} \sum_{j=1}^{n} a_{ij} q_j = \max_{p \in X^*} p^T A q$$

Hence II would choose q that minimize the above.

The minimum value is

$$\overline{V} = \min_{q \in Y^*} \max_{1 \leq i \leq m} \sum_{j=1}^{n} a_{ij} q_j = \min_{q \in Y^*} \max_{p \in X^*} p^T A q$$

This is called the **upper value** of the game. Any strategy $q \in Y^*$ that achieves this minimum is called a **minimax strategy** for II.

Similarly, suppose that I is required to announce his choice of $p \in X^*$.

Then II would use his Bayes strategy against p and I would win the following amount

$$\min_{1 \le j \le n} \sum_{i=1}^{m} p_i a_{ij} = \min_{q \in Y^*} p^T A q$$

Hence I would choose p that maximize the above.

The maximum value is

$$\underline{V} = \max_{p \in X^*} \min_{1 \le j \le n} \sum_{i=1}^{m} p_i a_{ij} = \max_{p \in X^*} \min_{q \in Y^*} p^T A q$$

This is called the **lower value** of the game. Any strategy $p \in X^*$ that achieves this maximum is called a **minimax strategy** for I.

Lemma: In a finite game, both players have minimax strategies.

Proof. q is minimax if q minimizes

$$\overline{V} = \min_{q \in Y^*} \max_{1 \le i \le m} \sum_{j=1}^{n} a_{ij} q_j = \min_{q \in Y^*} \max_{p \in X^*} p^T A q$$

But

$$\max_{1 \le i \le m} \sum_{j=1}^{n} a_{ij} q_j = \max_{p \in X^*} p^T A q$$

is the maximum of m linear functions of q, so it is a continuous function of q, and Y^* is a closed and bounded set. Hence the minimum is achieved.

Lemma: We have $\underline{V} \leq \overline{V}$.

Proof. This follows from the following general result.

$$\max_{x \in X^*} \min_{y \in Y^*} f(x,y) \leq \min_{y \in Y^*} \max_{x \in X^*} f(x,y)$$

Definition: If $\underline{V} = \overline{V}$, we say the value V of the game exists, and define $V = \underline{V} = \overline{V}$. If the value exists, the minimax strategies are called optimal strategies.

Theorem: (**The Minimax Theorem**) Every finite game has a value, and both players have optimal strategies.

(Proof is omitted.)

Lemma: Let A and A' are matrices with $a'_{ij} = ca_{ij} + b$ where $c > 0$. Then the two games have the same minimax strategies. Moreover, $V' = cV + b$.

Solving games by linear programming

Consider Player I. He wants to choose $p_1, \cdots, p_m$ to

$$\text{maximize} \quad \min_{1 \le j \le n} \sum_{i=1}^{m} p_i a_{ij}$$

subject to the constraints

$$p_1 + \cdots + p_m = 1, \quad p_i \ge 0.$$

But this is not a linear programming problem, since the objective function is not linear.

We can convert this into a linear programming problem by the following trick.

Let $v = \min_{1 \le j \le n} \sum_{i=1}^{m} p_i a_{ij}$.

Then we find v and $p_1, \cdots, p_m$ to

$$\text{maximize} \quad v$$

subject to the constraints

$$v \le \sum_{i=1}^{m} p_i a_{i1} \quad \cdots \quad v \le \sum_{i=1}^{m} p_i a_{in}$$

$$p_1 + \cdots + p_m = 1, \quad p_i \ge 0.$$

This is a linear programming problem since both the objective function and the constraints are linear.

Similarly, for Player II, we have the following linear programming

Then we find w and $q_1, \cdots, q_n$ to

$$\text{minimize} \quad w$$

subject to the constraints

$$w \geq \sum_{j=1}^{n} a_{1j} q_j \quad \cdots \quad w \geq \sum_{j=1}^{n} a_{mj} q_j$$

$$q_1 + \cdots + q_n = 1, \quad q_j \geq 0.$$

Remark: The Duality Theorem, from theory of linear programming, says that the above two problems (p.11 and p.12) have the same value. This is exactly the result of the Minimax Theorem.

To solve the game by the Simplex Method, we need to further simplify the above problems.

Consider Player I (p.11).

Assume that the value of the game is positive, i.e., $v > 0$.

Introduce new variables $x_i = p_i/v$.

Then the constraints $p_1 + \cdots + p_m = 1$ implies $x_1 + \cdots + x_m = 1/v$.

But maximizing v is equivalent to minimizing $1/v$.

Thus, the problem in p.11 is written as

$$\text{minimize} \quad x_1 + \cdots + x_m$$

subject to the constraints

$$1 \leq \sum_{i=1}^{m} x_i a_{i1}, \quad \cdots \quad 1 \leq \sum_{i=1}^{m} x_i a_{in}, \quad \text{and} \quad x_i \geq 0$$

Simplex Method

Step 1: Add a constant to the matrix so that the value is positive.

Step 2: Form a tableau

$$
\begin{array}{c|cccc|c}
 & y_1 & y_2 & \cdots & y_n & \\
\hline
x_1 & a_{11} & a_{12} & \cdots & a_{1n} & 1 \\
x_2 & a_{21} & a_{22} & \cdots & a_{2n} & 1 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
x_m & a_{m1} & a_{m2} & \cdots & a_{mn} & 1 \\
\hline
 & -1 & -1 & \cdots & -1 & 0
\end{array}
$$

Step 3: Choose a pivot in the interior of the tableau. Say row p column q, with these properties:

1. the last entry in column q, $a(m+1, q)$, must be negative
2. the pivot $a(p, q)$ is positive
3. the pivot row p must be chosen so that the ratio $a(p, n+1)/a(p, q)$ is smallest among all other pivots on the same column

Step 4: Pivot

1. $p \to 1/p$ ($p =$ pivot)
2. $r \to r/p$ ($r =$ all entries on the same row as pivot)
3. $c \to -c/p$ ($c =$ all entries on the same column as pivot)
4. $q \to q - (rc/p)$

Step 5: Exchange label of pivot row and column

Step 6: If there are any negative numbers in the last row, go back to Step 3

Step 7: Done

1. the value v is the reciprocal of the value in lower right corner

2. I's optimal strategy can be constructed as follows. Those variables remain on the left receive probability zero. Otherwise, the probability for a particular variable is the value on the last row divided by the value in lower right corner

3. II's optimal strategy can be constructed as follows. Those variables remain on the top receive probability zero. Otherwise, the probability for a particular variable is the value on the last column divided by the value in lower right corner

Example: Consider the matrix game

$$B = \begin{pmatrix} 2 & -1 & 6 \\ 0 & 1 & -1 \\ -2 & 2 & 1 \end{pmatrix}$$

Note: no saddle point nor domination

Is the value positive?

Adding 2 to the matrix, we have

$$B' = \begin{pmatrix} 4 & 1 & 8 \\ 2 & 3 & 1 \\ 0 & 4 & 3 \end{pmatrix}$$

The value is at least 1 (by choosing row 1 for example).

Step 2: Form a tableau

	y_1	y_2	y_3	
x_1	4	1	8	1
x_2	2	3	1	1
x_3	0	4	3	1
	-1	-1	-1	0

Step 3: Choose pivot. Note that all 3 columns have negative entries in the last row. We choose column 1 as the pivot column. To choose pivot row, we can use either row 1 or row 2 (since the third row is zero). The ratios of the last column to the pivot are 1/4 and 1/2 resp. Hence, we choose row 1 as pivot row.

Step 4: Pivot.

$$
\begin{array}{c|ccc|c}
 & y_1 & y_2 & y_3 & \\
\hline
x_1 & 4 & 1 & 8 & 1 \\
x_2 & 2 & 3 & 1 & 1 \\
x_3 & 0 & 4 & 3 & 1 \\
\hline
 & -1 & -1 & -1 & 0
\end{array}
\quad \longrightarrow \quad
\begin{array}{c|ccc|c}
 & x_1 & y_2 & y_3 & \\
\hline
y_1 & 1/4 & 1/4 & 2 & 1/4 \\
x_2 & -1/2 & 5/2 & -3 & 1/2 \\
x_3 & 0 & 4 & 3 & 1 \\
\hline
 & 1/4 & -3/4 & 1 & 1/4
\end{array}
$$

(To obtain the entry on row 2 column 3, we replace 1 by $1 - 8 \cdot 2/4 = -3$.)

Step 5: We have interchanged the labels x_1 and y_1.

Step 6: There is one negative entry on column 2. Go back Step 3.

Go back Step 3: Column 2 is pivot column. To choose pivot row, we observe that the ratios of the last column to column 2 are $1, 1/5, 1/4$. Thus row 2 is the pivot row.

Step 4 and 5:

	x_1	y_2	y_3	
y_1	$1/4$	$1/4$	2	$1/4$
x_2	$-1/2$	$5/2$	-3	$1/2$
x_3	0	4	3	1
	$1/4$	$-3/4$	1	$1/4$

$\rightarrow$

	x_1	x_2	y_3	
y_1	0.3	-0.1	2.3	0.2
y_2	-0.2	0.4	-1.2	0.2
x_3	0.8	-1.6	7.8	0.2
	0.1	0.3	0.1	0.4

Note: all entries on last row are non-negative. Go to Step 7.

Step 7: Read the solution.

	x_1	x_2	y_3	
y_1	0.3	−0.1	2.3	0.2
y_2	−0.2	0.4	−1.2	0.2
x_3	0.8	−1.6	7.8	0.2
	0.1	0.3	0.1	0.4

The value is $1/0.4 = 5/2$. The value of the original game is $v = 5/2 - 2 = 1/2$.

Since x_3 is still on the left, $p_3 = 0$. And $p_1 = 0.1/0.4 = 1/4$ and $p_2 = 0.3/0.4 = 3/4$. Thus, I's optimal strategy is $p = (1/4, 3/4, 0)$.

Since y_3 is still on the top, $q_3 = 0$. And $q_1 = 0.2/0.4 = 1/2$ and $q_2 = 0.2/0.4 = 1/2$. Thus, II's optimal strategy is $q = (1/2, 1/2, 0)$.

Chapter 2: Two-person zero-sum games

Section 2.5: The extensive form of a game

Game tree

The extensive form of a game is modeled using a directed graph.

A **directed graph** is a pair (T, F) where T is a nonempty set of vertices and F is a function of followers (i.e. for each x, $F(x)$ is a subset of followers of x).

The vertices are positions of a game, and $F(x)$ are those positions that can be reached from x in one move.

A **path** from a vertex t_0 to a vertex t_1 is a sequence of vertices $x_0, x_1, \cdots, x_n$ such that $x_0 = t_0$, $x_n = t_1$, and x_i is a follower of x_{i-1}.

Next we define tree.

Definition: A **tree** is a directed graph (T, F) in which there is a special vertex t_0, called the root or initial vertex, such that for every other vertex t, there is a unique path beginning at t_0 and ending at t.

Interpretation:

Game starts at the initial vertex.

Continue along one of the paths.

At terminal vertices, the rules of the game specify payoffs.

Some non-terminal vertices are assigned to Player I while some others are assigned to Player II. There are also some non-terminal vertices from which a chance move is made. (e.g. rolling a dice or dealing of cards)

Basic endgame in poker

- Both players put 1 dollar on table. The money on the table is called pot.
- Player I gets a card. It is a winning card with prob. 1/4 and a losing card with prob. 3/4. Player I hides this card from II.
- Player I then checks or bets.
  - If he checks, his card is inspected. If he has a winning card, he wins 1. Otherwise, he loses 1.
  - If he bets, he will put 2 more dollars on the table.
- If I bets, Player II must fold or call.
  - If II folds, he loses 1 dollar.
  - If II calls, he adds 2 more dollars. Then I's card is inspected. If I has winning card, he wins 3. Otherwise, he loses 3.

We can draw a tree for this game. (see next page)

There is only one feature missing from this figure.

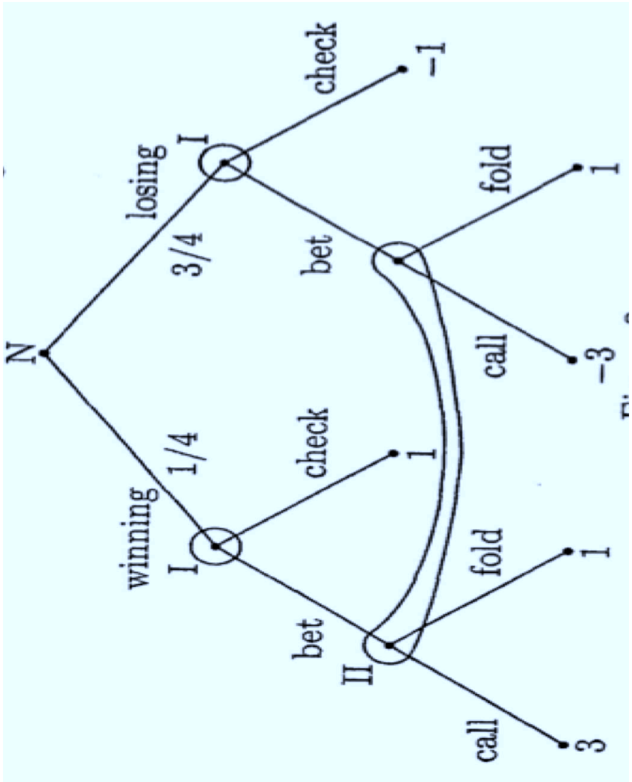
We have not indicated that at the time II makes his decision, he does not know which card I has received. That is, II does not know which of his two possible positions he is.

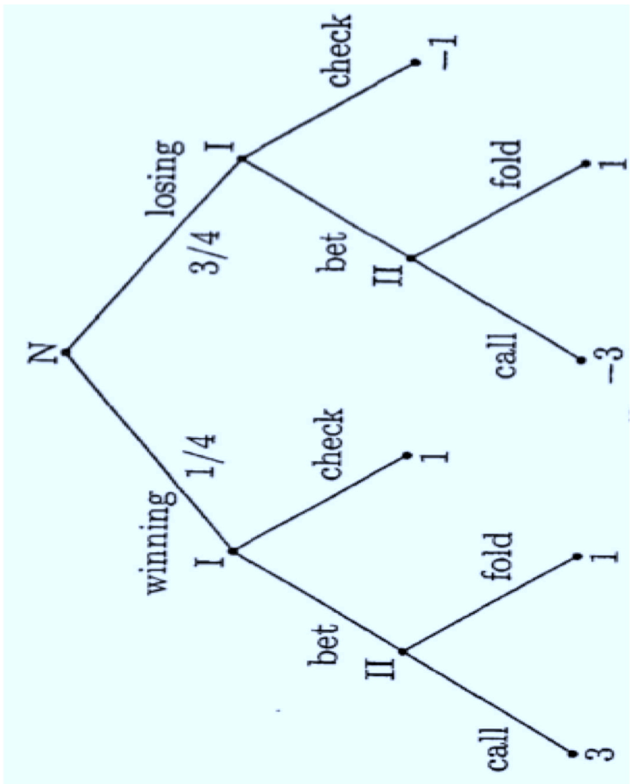
We indicate this by circling the two positions. (see next page)

We say that these two vertices form an **information set**.

The two vertices at which I has to move form two separate information sets, since he is told the outcome of the chance move. We use two circles to indicate this.

A tree with all payoffs, information sets, and labels of edges and vertices is known as the **Kuhn Tree**.


N
winning
losing
1/4
3/4
I
I
check
bet
check
bet
II
1
−1
call
fold
call
fold
3
1
−3
1


N
winning
losing
1/4
3/4
I
I
check
bet
check
bet
II
II
1
−1
call
fold
call
fold
3
1
−3
1

Represent Strategic form in Extensive form

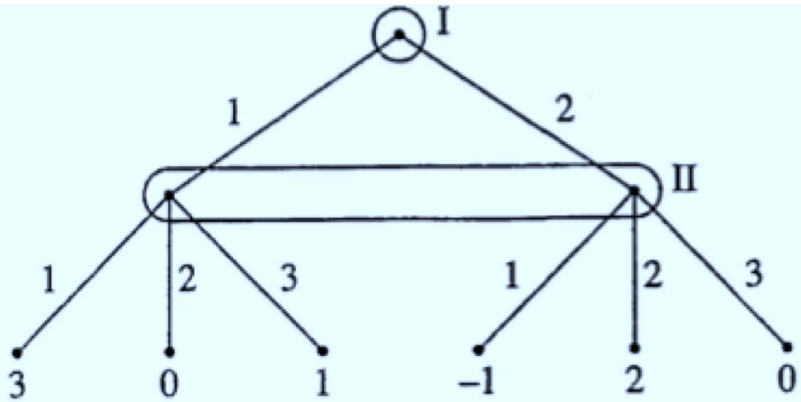
Consider the 2×3 matrix game in strategic form.

Note, in strategic form, players make simultaneous moves. However, in extensive form, moves are made sequentially.

We let Player I moves first. Then let Player II moves without knowing Player I's move. This may be described by the use of a suitable information set.

$$\begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}$$

Matrix Form



Equivalent Extensive Form

Reduction of extensive form to strategic form

Consider the basic endgame in poker. (see Page 27)

Player I has 2 information sets. In each set, he chooses one of the two options. Thus, there are 4 pure strategies for I. Denoted by

- (b, b): bet with winning or losing card
- (b, c): bet with winning and check with losing card
- (c, b): check with winning card and bet with losing card
- (c, c): check with winning or losing card

Let $X = \{(b, b), (b, c), (c, b), (c, c)\}$.

Player II has one information set. $Y = \{c, f\}$.

- c: if I bets, II calls
- f: if I bets, II folds

Now we find the payoff matrix. We consider average return.

The matrix is

$$\begin{array}{cc} & \begin{array}{cc} c & f \end{array} \\ \begin{array}{c} (b,b) \\ (b,c) \\ (c,b) \\ (c,c) \end{array} & \begin{pmatrix} -3/2 & 1 \\ 0 & -1/2 \\ -2 & 1 \\ -1/2 & -1/2 \end{pmatrix} \end{array}$$

To find the upper left entry, since I uses (b, b) and II uses c, I wins 3 with prob. $1/4$ and loses 3 with prob. $3/4$. Thus

$$A((b,b),c) = \frac{1}{4}(3) + \frac{3}{4}(-3) = -\frac{3}{2}$$

To solve the game, we observe that row 3 is dominated by row 1, and row 4 is dominated by row 2. The matrix becomes

$$\begin{pmatrix} -3/2 & 1 \\ 0 & -1/2 \end{pmatrix}$$

Solving $p = (1/6, 5/6)$, $q = (1/2, 1/2)$ and $V = -1/4$

For the original game: $p = (1/6, 5/6, 0, 0)$, $q = (1/2, 1/2)$

Note:

1. Never check with a winning card.

2. (b, b) is a bluffing strategy, bet with a losing card.

3. (b, c) is a honest strategy, bet with winning card and check with losing card.

MAT 4250: Lecture 6

Eric Chung

Chapter 3: Two-person General-sum games

Section 3.1: Bimatrix games

Strategic form

Two-person general-sum game is given by

- two sets X and Y of pure strategies
- two real-valued functions $u_1(x, y)$ and $u_2(x, y)$

  (if I chooses $x \in X$ and II chooses $y \in Y$, I receives $u_1(x, y)$ and II receives $u_2(x, y)$)

The strategic form can be represented by a matrix of ordered pairs, called **bimatrix**.

Each entry of the bimatrix has two components, the first component is I's payoff and the second component is II's payoff.

Example: consider the bimatrix

$$\begin{pmatrix} (1,4) & (2,0) & (-1,1) & (0,0) \\ (3,1) & (5,3) & (3,-2) & (4,4) \\ (0,5) & (-2,3) & (4,1) & (2,2) \end{pmatrix}$$

Player I has 3 pure strategies, and II has 4 pure strategies.

If I chooses row 3 and II chooses column 2, the corresponding entry in the bimatrix is $(-2,3)$. Thus, I loses 2 and II wins 3.

We sometimes represent the game using two matrices (A,B):

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 5 & 3 & 4 \\ 0 & -2 & 4 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 0 & 1 & 0 \\ 1 & 3 & -2 & 4 \\ 5 & 3 & 1 & 2 \end{pmatrix}$$

where A represents payoff to I and B represents payoff to II.

Extensive form

Similar to two-person zero-sum games. For example, see fig7.pdf

One can reduce this to strategic form.

I has two pure strategies $X = \{c, d\}$, and II has two pure strategies $Y = \{a, b\}$. The game matrix is

$$\begin{array}{cc} & \begin{array}{cc} \quad a \quad & \quad\quad b \quad \end{array} \\ \begin{array}{c} c \\ d \end{array} & \begin{pmatrix} (5/4, 0) & (2/4, 3/4) \\ (0, 2/4) & (3/4, 2/4) \end{pmatrix} \end{array}$$

(To compute the upper left entry,
$1/4\ (-1, 3) + 3/4\ (2, -1) = (5/4, 0)$)

- Analysis of two-person general-sum games is more complex. In this case, maximizing one's payoff is not equivalent to minimizing the other's payoff. In particular, the minimax theorem does not apply.
- The theory is divided into two classes: **noncooperative theory** and **cooperative theory**.
- In noncooperative theory, the players are unable to communicate before decisions are made. This leads to the concept of **strategic equilibrium**.
- In cooperative theory, players are allow to communicate before decisions are made. They can jointly agree to use certain strategies.
  - If the players make side-payments, it is called a **TU cooperative game** (TU = transferrable utility)
  - Otherwise, it is a **NTU cooperative game**

Safety levels

Consider a two-person general-sum game with matrices A and B.

Player I can win on average at least (Why?)

$$v_I = \max_p \min_j \sum_{i=1}^m p_i a_{ij} = \mathrm{Val}(A)$$

This is called the **safety level** for Player I.

(The number Val(A) is the value of game A when considered as a two-person zero-sum game.)

Player I can win this amount without considering II's payoff matrix.

Any strategy p that achieves the maximum above is called a **maxmin strategy**.

Similarly, the safety level for Player II is

$$v_{II} = \max_{q} \; \min_{i} \sum_{j=1}^{n} b_{ij} q_j = \mathrm{Val}(B^T)$$

And Player II can win at least by this amount.

(The number $\mathrm{Val}(B^T)$ is the value of game B^T when considered as a two-person zero-sum game. Note that the value is the winning of the row chooser.)

Any strategy q that achieves the maximum above is called a **maxmin strategy**.

Example: Consider the game

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$$

- I's maxmin strategy is $(3/4, 1/4)$ and $v_I = 3/2$

  For B, col.2 > col.1. II's maxmin strategy is $(0, 1)$, $v_{II} = 2$

  If they both use maxmin, then I wins $v_I = 3/2$ and II wins $3(3/4) + 2(1/4) = 11/4$

  This is good for II as II gets more than v_{II}

- If I sees II's payoff, I knows that II will always choose col.2.

  Thus, I will choose row 2 and get 3. And II gets 2.

- The payoff $(3, 2)$ is rather stable, since if each player believes the other is going to use the second strategy, he will use the second strategy. One example of **strategic equilibrium**.

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$$

- In TU cooperative theory,
  - the players may jointly agree on using row 2 and column 2, and receive total payoff 5.
  - However, they may need to discuss how to split the 5.
  - Player II has a threat to use column 1. In this case, I gets 0 and II gets 1.
- In NTU cooperative theory, no transfer of payoff is allowed. In this case, the payoffs are in noncomparable units. They may agree on other strategy (e.g. the $(1, 3)$ payoff).

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Chapter 3: Two-person General-sum games

Section 3.2: Non-cooperative games

Strategic equilibrium

Assumption: players cannot cooperate to attain higher payoffs, if communication is allowed, no binding agreements can be formed.

A finite n-person game in strategic form is given by

- n nonempty sets $X_1, \cdots, X_n$
  (X_i is the set of pure strategies for player i)
- n real-valued functions $u_1, \cdots, u_n$ defined on $X_1 \times \cdots \times X_n$
  ($u_i(x_1, \cdots, x_n)$ is the payoff to player i)

Definition: A vector of pure strategy $(x_1, \cdots, x_n)$ is called a pure strategic equilibrium (PSE) if for all $i = 1, \cdots, n$ and $x \in X_i$,

$$u_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) \geq u_i(x_1, \cdots, x_{i-1}, x, x_{i+1}, \cdots, x_n)$$

Recall: $(x_1, \cdots, x_n)$ is a PSE if for all $i = 1, \cdots, n$ and $x \in X_i$,

$$u_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) \geq u_i(x_1, \cdots, x_{i-1}, x, x_{i+1}, \cdots, x_n)$$

If you are player i, and all other players use their corresponding pure strategies, then the best you (player i) can do is to use x_i.

Such x_i is called a **best response** for player i to the strategy choices of the other players.

If communication is allowed and some informal agreement is made, it should be a strategic equilibrium. Since no binding agreement can be made, they will agree on a strategy in which no one can gain by unilaterally violate the agreement.

Examples: Consider two-person games

$$(1) \quad \begin{pmatrix} (3,3) & (0,0) \\ (0,0) & (5,5) \end{pmatrix} \qquad (2) \quad \begin{pmatrix} (3,3) & (4,3) \\ (3,4) & (5,5) \end{pmatrix}$$

In (1), row 1 - col 1 (denoted $\langle 1, 1\rangle$) is a PSE. If each player believes that the other will use this strategy, he will not change his strategy.

$\langle 2, 2\rangle$ is also a PSE. Both players prefer this as it gives higher payoff.

In (2), $\langle 1, 1\rangle$ is a PSE. However, no player will hurt if he changes strategy. If they both change, they will both be better off.

The PSE $\langle 1, 1\rangle$ is rather unstable.

Now we extend the definition to mixed strategies.

We define $\mathcal{P}_k$ as a set of probabilities with length k

$$\mathcal{P}_k = \left\{ \mathbf{p} = (p_1, \cdots, p_k) \mid p_i \geq 0,\ \sum_{i=1}^{k} p_i = 1 \right\}$$

Let m_i be the number of elements in X_i.

Let X_i^* be the set of mixed strategies for player i. Then $X_i^* = \mathcal{P}_{m_i}$.

Denote $X_i = \{1, 2, \cdots, m_i\}$.

Suppose Player i uses his mixed strategy $\mathbf{p}_i = (p_1^{(i)}, \cdots, p_{m_i}^{(i)}) \in X_i^*$.

Then the average payoff to Player j is

$$g_j(\mathbf{p}_1, \cdots, \mathbf{p}_n) = \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} p_{i_1}^{(1)} \cdots p_{i_n}^{(n)}\ u_j(i_1, \cdots, i_n)$$

Definition: A vector of mixed strategies $(\mathbf{p}_1, \cdots, \mathbf{p}_n)$, $\mathbf{p}_i \in X_i^*$, is called a strategic equilibrium (SE) if for all i and $\mathbf{p} \in X_i^*$

$$g_i(\mathbf{p}_1, \cdots, \mathbf{p}_{i-1}, \mathbf{p}_i, \mathbf{p}_{i+1}, \cdots, \mathbf{p}_n) \geq g_i(\mathbf{p}_1, \cdots, \mathbf{p}_{i-1}, \mathbf{p}, \mathbf{p}_{i+1}, \cdots, \mathbf{p}_n)$$

The $\mathbf{p}_i$ is called a **best response** of Player i to the mixed strategies of other players.

No player can gain by unilaterally changing strategy.

Note that a PSE is a special case of SE.

Question: does SE always exist?

Theorem: (**Nash**) Every n-person game in strategic form has at least one strategic equilibrium. SE is also called Nash equilibrium.

Example: Consider the bimatrix game

$$\begin{pmatrix} (3,3) & (0,2) \\ (2,1) & (5,5) \end{pmatrix} \quad \text{that is} \quad A = \begin{pmatrix} 3 & 0 \\ 2 & 5 \end{pmatrix}, B = \begin{pmatrix} 3 & 2 \\ 1 & 5 \end{pmatrix}$$

- maxmin for I is $(1/2, 1/2)$ and safety level $v_I = 5/2$
- maxmin for II is $(3/5, 2/5)$ and safety level $v_{II} = 13/5$
- There are 2 PSEs, which are $\langle 1, 1 \rangle$ and $\langle 2, 2 \rangle$
- Consider $\langle 1, 1 \rangle$. If each player believes the other is going to use this, then they will use this. Otherwise, if one tries to change, it will actually hurt himself (by getting less payoff).
- Similar for $\langle 2, 2 \rangle$. If they can communicate, they will choose this because both get better payoff.

Refer to the last example, there is one more SE.

$$\begin{pmatrix} (3,3) & (0,2) \\ (2,1) & (5,5) \end{pmatrix} \quad \text{that is} \quad A = \begin{pmatrix} 3 & 0 \\ 2 & 5 \end{pmatrix}, B = \begin{pmatrix} 3 & 2 \\ 1 & 5 \end{pmatrix}$$

Each player chooses an equalizing strategy using other's payoff matrix. This pair of mixed strategies form a SE.

Both players receive the same payoff no matter what the other does.

I has the equalizing strategy $p = (4/5, 1/5)$ using B.

II has the equalizing strategy $q = (5/6, 1/6)$ using A.

If both use this strategy, the payoff is $(5/2, 13/5)$.

This SE is extremely unstable. No one can gain by unilaterally changing strategy. But it does not harm for both players to change to another strategy.

Note that both players have same preference to all SE.

Example: The battle of the Sexes.

$$\begin{pmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{pmatrix} \quad \text{that is} \quad A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Husband and wife are choosing which movie, 1 or 2, to see.

They prefer different movies, but going together is preferable to going alone.

$\langle 1,1 \rangle$ and $\langle 2,2 \rangle$ are both PSE. Player I prefers the first and Player II prefers the second.

The maxmin strategies are $(1/3, 2/3)$ and $(2/3, 1/3)$ for I and II resp. with safety levels $(v_I, v_{II}) = (2/3, 2/3)$.

The SE found by equalizing strategies is $p = (2/3, 1/3)$ and $q = (1/3, 2/3)$ with payoff $(2/3, 2/3)$. This is not a good choice for both players.

Example: The Prisoner's Dilemma

$$\begin{array}{cc} & \begin{array}{cc} \text{confess} & \text{silent} \end{array} \\ \begin{array}{c} \text{confess} \\ \text{silent} \end{array} & \begin{pmatrix} (3,3) & (0,4) \\ (4,0) & (1,1) \end{pmatrix} \end{array}$$

- Two criminals are captured and separated into different rooms.
- If one confesses and the other remains silent, the one who remain silent will be set free and the other will be sent to jail for maximum sentence.
- If both remain silent, they will be sent to jail for minimum sentence.
- If both confess, they can only be convicted for a very minor charge.

$$\begin{array}{cc} & \begin{array}{cc} \text{confess} & \text{silent} \end{array} \\ \begin{array}{c} \text{confess} \\ \text{silent} \end{array} & \begin{pmatrix} (3,3) & (0,4) \\ (4,0) & (1,1) \end{pmatrix} \end{array}$$

For player I, row 2 dominates row 1. He will remain silent.

For player II, col. 2 dominates col. 1. He will remain silent too.

(Note: $\langle 2, 2 \rangle$ is a PSE.)

Hence, they will receive the payoff $(1, 1)$ (they will be sent to jail for minimum sentence).

However, if they both use their dominated strategies, both get the payoff $(3, 3)$ (that is, they are convicted for a minor charge).

Thus, they are better off if they choose dominated strategies.

Some remarks:

In noncooperative game theory,

1. there are usually many different equilibria giving different payoffs.
2. if there exists a unique equilibrium it may not be considered as a reasonable solution.

Finding PSE

Idea:

- Put a ”star” to I’s payoffs that are max of the column.
- Put a ”star” to II’s payoffs that are max of the row.
- Then the entries with ”two star” are PSE.

Example:

$$\begin{pmatrix} (2,1) & (4,3) & (7^*,2) & (7^*,4) & (0,5^*) & (3,2) \\ (4^*,0) & (5^*,4) & (1,6^*) & (0,4) & (0,3) & (5^*,1) \\ (1,3^*) & (5^*,3^*) & (3,2) & (4,1) & (1^*,0) & (4,3^*) \\ (4^*,3) & (2,5^*) & (4,0) & (1,0) & (1^*,5^*) & (2,1) \end{pmatrix}$$

The PSE are $\langle 3,2\rangle$ and $\langle 4,5\rangle$.

MAT 4250: Lecture 7

Eric Chung

Chapter 3: Two-person General-sum games

Section 3.3: Models of Duopoly

The Cournot Model of Duopoly

Two competing firms are producing a product. (Assume making decision simultaneously.)

The price for producing one unit of the product is c.

Let q_i $(i = 1, 2)$ be the number of units produced by Firm i.

Let $Q = q_1 + q_2$ be the total number of units in the market.

We assume the following price function. (assume $c < a$)

$$P(Q) = \begin{cases} a - Q & \text{if } 0 \leq Q \leq a \\ 0 & \text{if } Q > a \end{cases} = (a - Q)^+$$

Let $X = Y = [0, \infty]$ be the set of pure strategies, and define the payoff functions

$$\begin{aligned} u_1(q_1, q_2) &= q_1 P(q_1 + q_2) - cq_1, \quad (q_1, q_2) \in X \times Y \\ u_2(q_1, q_2) &= q_2 P(q_1 + q_2) - cq_2, \quad (q_1, q_2) \in X \times Y \end{aligned}$$

First, we consider the monopoly case, that is, $q_2 = 0$.

The payoff is $u(q_1) = q_1(a - q_1)^+ - cq_1$.

Note $u(q_1)$ will be positive for $0 < q_1 < a$.

Thus $u(q_1) = q_1(a - q_1) - cq_1 = q_1(a - c) - q_1^2$ and $u'(q_1) = a - c - 2q_1$.

We see that the max of u is attained at $q_1 = (a - c)/2$ and the max value of u is $u((a - c)/2) = (a - c)^2/4$.

Thus, Firm I should make $(a - c)/2$ units of the product for a maximum profit of $(a - c)^2/4$.

Note that the corresponding price of the product is $P((a - c)/2) = a - (a - c)/2 = (a + c)/2$.

Now, we consider the duopoly case. We will find PSE.

We find q_1 and q_2 such that

$$\begin{aligned}\frac{\partial}{\partial q_1} u_1(q_1, q_2) &= a - 2q_1 - q_2 - c = 0 \\ \frac{\partial}{\partial q_2} u_2(q_1, q_2) &= a - q_1 - 2q_2 - c = 0\end{aligned}$$

Solving, we get $q_1^* = (a-c)/3$ and $q_2^* = (a-c)/3$.

(Note: if II uses q_2^*, the best I can do is to use q_1^*. And vice versa.)

Hence (q_1^*, q_2^*) is a PSE.

The corresponding profit for the Firms is $u_1(q_1^*, q_2^*) = (a-c)^2/9$.

The duopoly price is $P(q_1^*, q_2^*) = (a+2c)/3$, which is less than the monopoly price $(a+c)/2$. Consumers are better off in duopoly.

In duopoly, each Firm receives $(a-c)^2/9$. The total is $2(a-c)^2/9$.

Recall that in monopoly, the Firm receives $(a-c)^2/4$. This amount is greater than the total amount in duopoly case.

Thus, if the two Firms are allowed to communicate, they can improve their profits by agreeing share the production and profits.

They will each produce $(a-c)/4$ units and receive $(a-c)^2/8$ profit. In this case, they produce less units and receive more profit.

The Stackelberg Model of Duopoly

In this model, one player (the dominant player) moves first and let the other knows the outcome, and then the second player moves.

Assume that Firm I produces q_1 units.

Firm II needs to find q_2 to maximize its profit. To do so, Firm II finds q_2 by solving

$$\frac{\partial}{\partial q_2} u_2(q_1, q_2) = a - q_1 - 2q_2 - c = 0$$

Thus $q_2(q_1) = (a - q_1 - c)/2$.

Now Firm I knows this, and the payoff function becomes a function of q_1 only, i.e.

$$u_1(q_1, q_2(q_1)) = q_1(a - q_1 - \frac{a - q_1 - c}{2}) - cq_1$$

Simplifying, we get

$$u_1(q_1, q_2(q_1)) = -\frac{1}{2}q_1^2 + \frac{a-c}{2}q_1$$

I will find q_1 to maximize its profit. Thus it will find q_1 by solving

$$\frac{\partial}{\partial q_1}u_1(q_1, q_2(q_1)) = -q_1 + \frac{a-c}{2} = 0$$

So, $q_1^* = (a-c)/2$. This imples $q_2^* = (a-c)/4$.

- (q_1^*, q_2^*) is a SE.
- Firm I produces the monopoly quantity, and II produces less than Cournot SE.
- I's proft is $(a-c)^2/8$ which is greater than the Cournot SE profit, and II's profit is $(a-c)^2/16$ which is less than the Cournot SE profit.

  (Note that the information given out by I helps the firm to increase its profit.)
- Note that the total number of units produced in the Stackelbreg and Cournot models are $3(a-c)/4$ and $2(a-c)/3$ resp. This implies that the price under the Stackelbreg model is lower, and consumers are better off.

Entry deterrence

Consider a monopolist in a market. Sometimes, there are reasons for the firm to charge less than the monopoly price.

One of the reasons is that the high price of the product may attract another firm to enter the market.

Suppose the price/demand function is given by

$$P(Q) = \begin{cases} 17 - Q & \text{if } 0 \le Q \le 17 \\ 0 & \text{if } Q > 17 \end{cases}$$

and the cost for producing q_1 units is $q_1 + 9$.

The profit for the firm is $u_1 = (17 - q_1)q_1 - (q_1 + 9) = 16q_1 - q_1^2 - 9$.

Thus, the monopoly quantity is 8, the monopoly price is 9 and the monopoly profit is 55.

Now a competing firm wants to enter the market by producing q_2.

Assume that the cost is the same, namely, $q_2 + 9$.

Then the price will drop to $P(8 + q_2) = 9 - q_2$.

The profit for the competing firm is
$u_2 = (9 - q_2)q_2 - (q_2 + 9) = 8q_2 - q_2^2 - 9$.

The max profit is $u_2 = 7$ with $q_2 = 4$. Thus the firm has incentive to enter the market.

In this case, the price of the product is $P(8 + 4) = 5$.

The original monopolist's profit becomes $5 \cdot 8 - 17 = 23$ (compared to the monopoly profit 55).

Hence, the monopolist should do something to stop the competing firm to enter the market.

The monopolist can produce more units to deter the competing firm to enter the market.

Assume the monopolist produces q_1 units.

Then the profit of the competing firm is

$$u_2 = (17 - q_1 - q_2)q_2 - (q_2 + 9)$$

The max profit for the competing firm is $(16 - q_1)^2/4 - 9$ with $q_2 = (16 - q_1)/2$.

Thus, the competing firm will have no profit if the monopolist produces $q_1 = 10$. In this case, the competing firm has no incentive to enter the market.

For the monopolist, by producing $q_1 = 10$, the profit becomes $(17 - 10)10 - 19 = 51$. (compared to the monopoly profit 55)

There is a small price to pay to deter competing firm.

Chapter 3: Two-person General-sum games

Section 3.4: Cooperative games

Feasible set of payoff vectors

Consider the bimatrix game with $m \times n$ matrices (A, B).

In cooperative games, players can jointly agree on using one of the mn strategies, or even a probability mixture of these mn strategies.

In NTU games, transfer of utility is not allowed.

Hence the players can achieve one of the mn payoff vectors (a_{ij}, b_{ij}) or a probability mixture of these mn payoff vectors.

The set of all such payoff vectors is called a NTU feasible set.

Definition: The **NTU feasible set** is the convex hull of the mn points (a_{ij}, b_{ij}), for all $1 \leq i \leq m$, $1 \leq j \leq n$.

In TU games, transfer of utility is allowed.

By making a side payment, the payoff vector (a_{ij}, b_{ij}) can be changed to $(a_{ij} + s, b_{ij} - s)$.

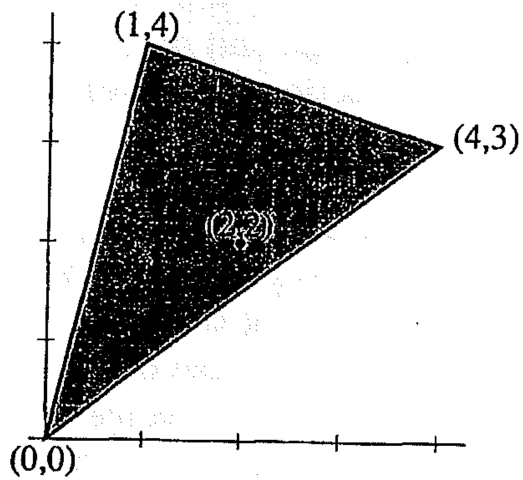
If $s > 0$, this represents payment from II to I. If $s < 0$, this represents payment from I to II.

Hence, the straight line through the point (a_{ij}, b_{ij}) with slope -1 are all possible payoff vectors.

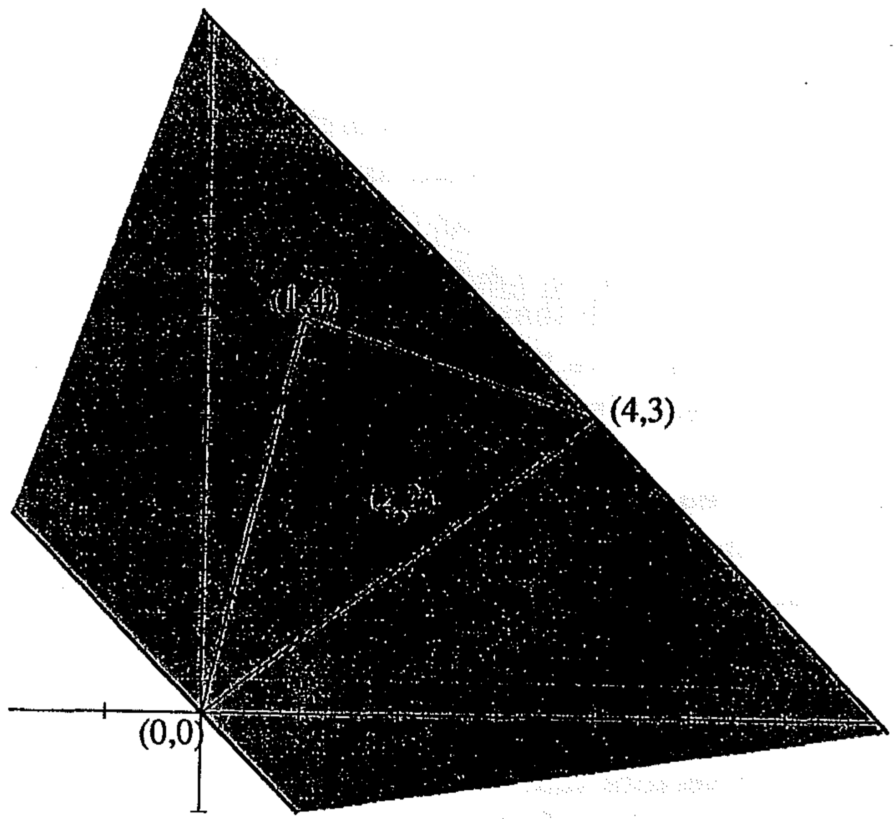
Definition: The **TU feasible set** is the convex hull of the all points in the form $(a_{ij} + s, b_{ij} - s)$ for all $1 \leq i \leq m$, $1 \leq j \leq n$, and all real number s.

Example: find NTU and TU feasible sets for

$$\begin{pmatrix} (4,3) & (0,0) \\ (2,2) & (1,4) \end{pmatrix}$$



The NTU Feasible Set



The TU Feasible Set

In cooperative game, it is expected that no player can be made better off without making the other player worse off.

In other words, the players cannot have other strategies that gives better payoff to at least one player.

This is the concept of Pareto optimality.

Definition: A feasible payoff vector (v_1, v_2) is said to be **Pareto optimal** if the only feasible vector (v_1', v_2') such that $v_1' \geq v_1$ and $v_2' \geq v_2$ is $(v_1', v_2') = (v_1, v_2)$.

Example: what are the Pareto optimal feasible payoff vectors for

$$\begin{pmatrix} (4,3) & (0,0) \\ (2,2) & (1,4) \end{pmatrix}$$

In general, the Pareto optimal feasible payoff vectors is the set of upper right boundary points.

TU cooperative games

In TU games, players will discuss the possibility of choosing a joint strategy, together with some possible side payments.

They also discuss what will happen if they cannot come to an agreement. Each may threaten to use some unilateral strategy what is bad for the opponent.

If they come to an agreement, we assume that the payoff vector is Pareto optimal.

They may make some threat. However, the thread must not hurt the player who makes it to a greater degree than the opponent.

They also decide how the payoff is divided among them. Here we assume that the two player's payoffs are measured in the same unit.

Example: Consider the bimatrix game

$$\begin{pmatrix} (5,3) & (0,-4) \\ (0,0) & (3,6) \end{pmatrix}$$

Players come to an agreement to use row 2 and col 2 because it has the largest total payoff.

They need to discuss how to divide the payoff.

Player I may threaten to use row 1 if he is not given at least 5.

Player II cannot make the counter threat to use col 2 because this will hurt II more.

Q: How to choose threats and side payments?

Solving TU games

The player will achieve the largest possible payoff σ

$$\sigma = \max_i \max_j (a_{ij} + b_{ij})$$

and then discuss how to divide this payoff.

They will choose some row i_0 and column j_0 such that $a_{i_0 j_0} + b_{i_0 j_0} = \sigma$. This strategy $\langle i_0, j_0 \rangle$ is called their **cooperative strategy**.

They must also decide how to divide the payoff. That is, they need to find some payoff vector (x^*, y^*) such that $x^* + y^* = \sigma$.

If $x^* > a_{i_0 j_0}$, then II needs to give side payment of $x^* - a_{i_0 j_0}$ to I.

If $x^* < a_{i_0 j_0}$, then I needs to give side payment of $a_{i_0 j_0} - x^*$ to II.

Suppose that I and II have **threat strategies** p and q.

Threat strategies are strategies that the two players will use if they cannot come to an agreement.

If they cannot come to an agreement, I will get $p^T Aq$ and II will get $p^T Bq$. (these are the average payoffs.)

Define $(D_1, D_2) = (p^T Aq, p^T Bq)$.

Note (D_1, D_2) is a point in the NTU feasible set.

(D_1, D_2) is called the **disagreement point** or **threat point**.

Now, the players need to determine (x^*, y^*) on the line $x + y = \sigma$.

Note I will not accept any amount less than D_1, and II will not accept any amount less than D_2.

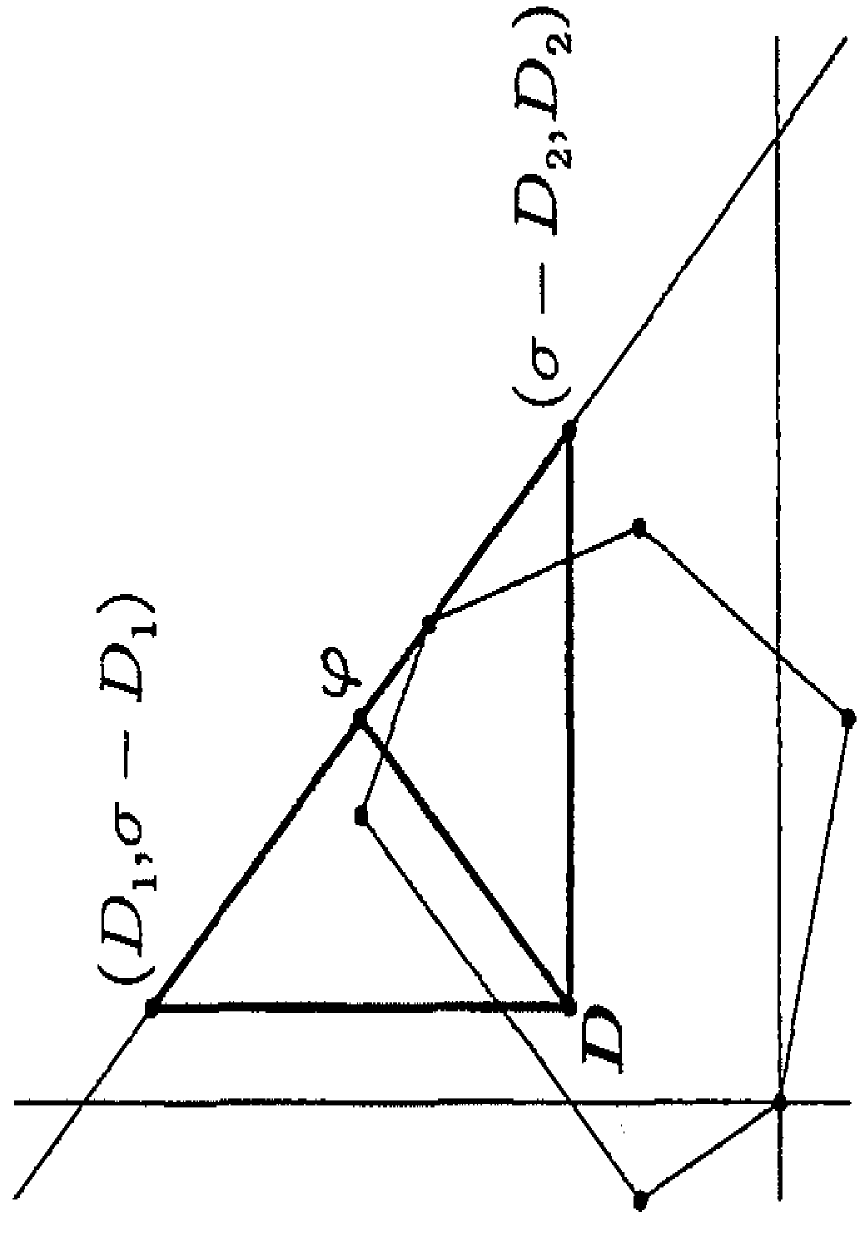
Thus, they need to find a point within the line segment from $(D_1, \sigma - D_1)$ to $(\sigma - D_2, D_2)$.

Clearly, the midpoint of this line segment

$$\phi = (\phi_1, \phi_2) = (\frac{\sigma - D_2 + D_1}{2}, \frac{\sigma - D_1 + D_2}{2})$$

is a natural choice. And both suffer equally it they break the agreement.

To complete the solution, we need to find p and q.


$(D_1, \sigma - D_1)$
$(\sigma - D_2, D_2)$
φ
D

From

$$\phi = (\phi_1, \phi_2) = (\frac{\sigma - D_2 + D_1}{2}, \frac{\sigma - D_1 + D_2}{2})$$

we see that

- I wants to maximize $D_1 - D_2$.
- II wants to minimize $D_1 - D_2$.

This is a two-person zero-sum game with matrix $A - B$.

Let p^* and q^* be the optimal strategies and δ be the value. Then

$$\delta = (p^*)^T(A - B)(q^*) = (p^*)^T A(q^*) - (p^*)^T B(q^*)$$

Take threat strategies p^* and q^*. Then $(D_1^*, D_2^*) = ((p^*)^T Aq^*, (p^*)^T Bq^*)$. The corresponding ϕ^* is

$$\phi^* = (\phi_1^*, \phi_2^*) = (\frac{\sigma + \delta}{2}, \frac{\sigma - \delta}{2})$$

Example: Consider the TU game with matrix

$$\begin{pmatrix} (0,0) & (6,2) & (-1,2) \\ (4,-1) & (3,6) & (5,5) \end{pmatrix}$$

The PSE is at $\langle 1,2\rangle$ with payoff $(6,2)$.

Now we find the TU solution. The maximum of $a_{ij}+b_{ij}$ occurs at the second row and third column with $\sigma = 10$.

Thus the TU cooperative strategy is $\langle 2,3\rangle$.

To find the threat strategies and side payment, consider the zero-sum game

$$A - B = \begin{pmatrix} 0 & 4 & -3 \\ 5 & -3 & 0 \end{pmatrix}$$

Note: the first col is dominated by the third col.

Solving this game, we get

$$p^* = (0.3, 0.7), \quad q^* = (0, 0.3, 0.7), \quad \delta = -0.9$$

Thus, the TU payoff is

$$\phi^* = ((10 - 0.9)/2, (10 + 0.9/2)) = (4.55, 5.45)$$

Hence, I has to give 0.45 side payment to II.

Moreover, the threat point $D^* = (D_1^*, D_2^*)$ is

$$D_1^* = (p^*)^T A q^* = 3.41, \quad D_2^* = (p^*)^T B q^* = 4.31$$

Example: Consider the TU game with matrix

$$\begin{pmatrix} (1,5) & (2,2) & (0,1) \\ (4,2) & (1,0) & (2,1) \\ (5,0) & (2,3) & (0,0) \end{pmatrix}$$

There are two cooperative strategies $\langle 1,1 \rangle$ and $\langle 2,1 \rangle$ giving total payoff $\sigma = 6$.

Consider the zero-sum game

$$A - B = \begin{pmatrix} -4 & 0 & -1 \\ 2 & 1 & 1 \\ 5 & -1 & 0 \end{pmatrix}$$

There are two saddle points $\langle 2,3 \rangle$ and $\langle 2,2 \rangle$ giving value $\delta = 1$. Thus there are two possible threat strategies.

$$\begin{pmatrix} (1,5) & (2,2) & (0,1) \\ (4,2) & (1,0) & (2,1) \\ (5,0) & (2,3) & (0,0) \end{pmatrix}$$

For the saddle point $\langle 2, 3 \rangle$, we have

$$p^* = (0,1,0), \quad q^* = (0,0,1)$$

Thus, $D^* = (D_1^*, D_2^*) = (2,1)$ and $\phi^* = (7/2, 5/2)$.

For the saddle point $\langle 2, 2 \rangle$, we have

$$p^* = (0,1,0), \quad q^* = (0,1,0)$$

Thus, $D^* = (D_1^*, D_2^*) = (1,0)$ and $\phi^* = (7/2, 5/2)$.

Note: same TU payoff in both cases.

Note: the side payment is determined by the choice of the cooperative strategy.

NTU games

In NTU games, side payments are not allowed.

It is assumed that payoffs are in noncomparable units.

The players can also threaten the other, and come to an agreement.

We will consider the **Nash Bargaining Model**.

It consists of two elements:

- A compact and convex set S
- A point $(u^*, v^*) \in S$

We think S is the NTU feasible set and(u^*, v^*) is a threat point.

Then we need to find a point $(\overline{u}, \overline{v}) = f(S, u^*, v^*)$ such that it is considered as a "reasonable solution" to the NTU game.

Nash has a few axioms to define "reasonable solution".

(1) **Feasibility**. $(\overline{u}, \overline{v}) \in S$.

(2) **Pareto optimality**.

(3) **Symmetry**. If S is symmetric about the line $u = v$ and $u^* = v^*$, then $\overline{u} = \overline{v}$.

(If the game is symmetric in the players, the solution should also be symmetric.)

(4) **Independence of irrelevant alternatives**. If T is a closed and convex subset of S, and if $(u^*, v^*) \in T$ and $(\overline{u}, \overline{v}) \in T$, then $f(T, u^*, v^*) = (\overline{u}, \overline{v})$.

(If the solution is inside T, then anything outside T are irrelevant.)

(5) **Invariance under change of location and scale**. If $T = \{(u', v') \mid u' = \alpha_1 u + \beta_1, v' = \alpha_2 v + \beta_2\}$ with $\alpha_1, \alpha_2 > 0$, then $f(T, \alpha_1 u^* + \beta_1, \alpha_2 v^* + \beta_2) = (\alpha_1 \overline{u} + \beta_1, \alpha_2 \overline{v} + \beta_2)$.

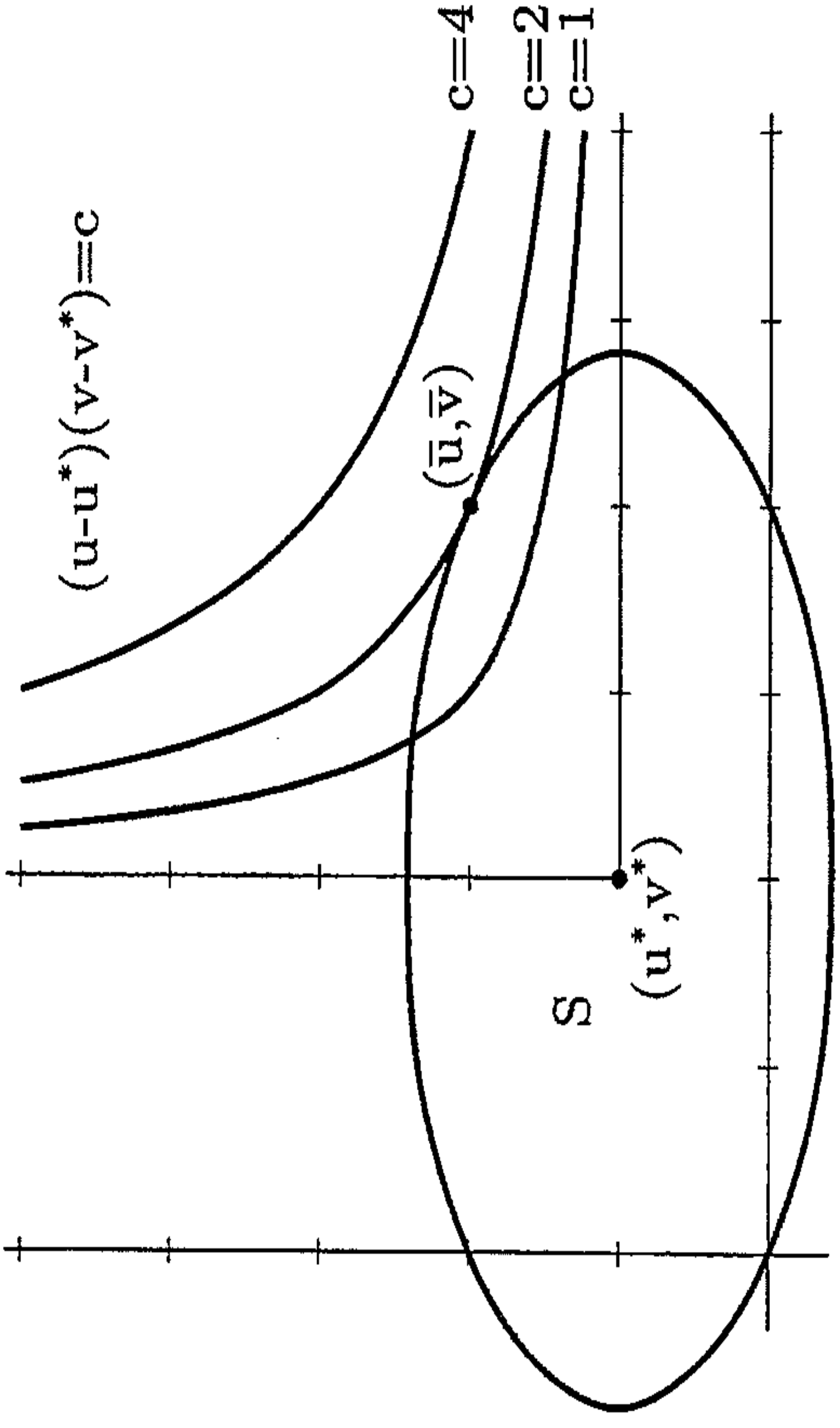
Theorem: There exists a unique function f satisfying the Nash axioms. Moreover, if there exists a point $(u, v) \in S$ such that $u > u^*, v > v^*$, then $(\overline{u}, \overline{v})$ maximizes the function $(u - u^*)(v - v^*)$ over the set of points with $u \geq u^*, v \geq v^*$.

Geometric interpretation:

Consider the family of curves $(u - u^*)(v - v^*) = c$.

One of these curves will "touch" the set S at one point. And this point is exactly $(\overline{u}, \overline{v})$.

Note also that the slope of this curve at the point $(\overline{u}, \overline{v})$ is the negative of the slope of the line connecting $(\overline{u}, \overline{v})$ and (u^*, v^*).



$(u-u^*)(v-v^*)=c$
c=4
c=2
c=1
$(\bar{u},\bar{v})$
S
(u^*,v^*)

Example: Let S be a triangle with vertices $(0,0)$, $(0,1)$ and $(3,0)$.

Let the threat point $(u^*, v^*) = (0,0)$.

The set of Pareto optimal points is the line from $(0,1)$ to $(3,0)$.

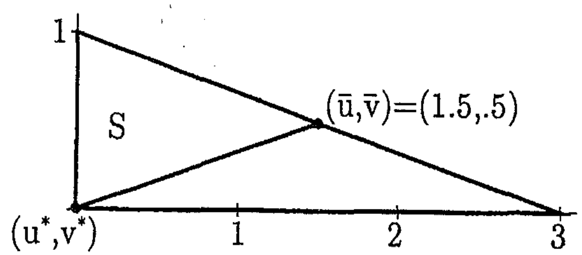
This line has slope $-1/3$.

The slope of the curve $uv = c$ at $(\overline{u}, \overline{v})$ has slope $-1/3$

Thus the slope of the line connecting $(0,0)$ and $(\overline{u}, \overline{v})$ has slope $1/3$.

Hence $(\overline{u}, \overline{v}) = (3/2, 1/2)$.

This is the NTU solution.



λ-transfer game

We will discuss the **λ-transfer** method. Consider the NTU game with matrices (A, B)

- Assume one unit of I's payoff is λ unit of II's payoff ($\lambda > 0$)
- Find the TU-solution for the game $(\lambda A, B)$.
- Divide I's payoff by λ. The payoff is $(\phi_1^*/\lambda, \phi_2^*)$
- If $(\phi_1^*/\lambda, \phi_2^*)$ is in the NTU feasible set of (A, B), then we take this as the NTU solution.

The TU solution to the game $(\lambda A, B)$ is

$$\sigma(\lambda) = \max_{i\ j}(\lambda a_{ij} + b_{ij}), \quad \delta(\lambda) = \mathrm{Val}(\lambda A - B)$$

The NTU payoff is then

$$\phi(\lambda) = (\phi_1(\lambda), \phi_2(\lambda)) = (\frac{\sigma(\lambda) + \delta(\lambda)}{2\lambda}, \frac{\sigma(\lambda) - \delta(\lambda)}{2})$$

if this point lies in the NTU feasible set of (A, B).

In general, there exists a unique λ^* with this property. This λ^* is called the **equilibrium exchange rate**.

The corresponding payoff $\phi(\lambda^*)$ is used as NTU solution.

Note: λ^* may be difficult to find.

Fixed threat point game

Consider the NTU bimatrix game (A, B).

Assume A and $-B$ have saddle points in the same position.

This game is called a **fixed threat point game**.

In this case, the zero-sum game $\lambda A - B$ has a saddle point at the same location. Thus, this game is very easy to solve.

Note that the threat strategies are independent of λ and threat point is easy to find.

Example: Consider the bimatrix game

$$\begin{pmatrix} (-1,1) & (1,3) \\ (0,0) & (3,-1) \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 1 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}$$

Clearly, A and $-B$ have saddle point at $\langle 2, 1 \rangle$.

Hence the matrix

$$\lambda A - B = \begin{pmatrix} -\lambda - 1 & \lambda - 3 \\ 0 & 3\lambda + 1 \end{pmatrix}$$

also has a saddle point at $\langle 2, 1 \rangle$.

Thus the threat strategies are $p^* = (0, 1)$ and $q^* = (1, 0)$.

And the value is $\delta(\lambda) = 0$. The threat point is $(0, 0)$.

Now we apply the Nash method with threat point $(0, 0)$.

The Pareto feasible set are the line from $(1, 3)$ to $(3, -1)$.

This line has slope -2.

So, the line from $(0, 0)$ to $(\overline{u}, \overline{v})$ has slope 2.

Hence $(\overline{u}, \overline{v}) = (1.25, 2.5)$.

From the definition of ϕ, we see that $\lambda^* = 2.5/1.25 = 2$.

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MAT 4250: Lecture 8

Eric Chung

Chapter 4: Games in coalitional form

Section 4.1: Many-person TU games

Introduction

We consider many-person TU cooperative games.

Agreements can be made among players.

Payoffs are measured in same unit, side payment is allowed.

Side payments may be used as incentives for some players to use certain mutually beneficial strategies.

Thus, there is tendency for players with similar objectives to form coalitions.

Coalitional form

Let n be the number of players, $n \geq 2$.

Let $N = \{1, 2, \cdots n\}$ be the set of players.

A **coalition** is defined as a subset S of N.

The set of all coalitions is denoted by 2^N.

The empty set ϕ is called the **empty coaltion**, and the set N is called the **grand coalition**.

For example, if $n = 2$, the set of all coalitions is $\{\phi, \{1\}, \{2\}, \{1, 2\}\}$.

If $n = 3$, the set of all coalitions is
$\{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, N\}$.

In general, the number of elements in 2^N is 2^n.

Definition: The **coalitional form** is a n-person game is given by the pair (N, v) where v is a real-valued function, called the **characteristic function** of the game, defined on 2^N satisfying

1. $v(\phi) = 0$
2. if S and T are disjoint coalitions ($S \cap T = \phi$), then $v(S) + v(T) \leq v(S \cup T)$

Remark:

(a). The quantity $v(S)$ is considered as the value of coalition S.

(b). Condition 2 says that when two disjoint coalitions work together, the value should be at least as much as the amount when they work apart.

Relation to strategic form

Strategic form: (X_i, u_i), $i = 1, 2, \cdots n$

Transforming to coalitional form, we need to define $v(S)$ for each S.

$v(S)$ is defined as the value of the two-person zero-sum game when S is considered as one player, and $\overline{S} = N - S$ is considered as the other player.

The payoff function for players in S is $\sum_{i \in S} u_i(x_1, \cdots, x_n)$.

This is the analogue of the safety level.

$v(S)$ represents the amount S can get without considering the actions of the players in $\overline{S}$.

Example: Consider the 3-person game, each has 2 pure strategies

If I chooses "1",

$$\begin{array}{cc} & \text{III} \\ \text{II} & \begin{pmatrix} (0,3,1) & (2,1,1) \\ (4,2,3) & (1,0,0) \end{pmatrix} \end{array}$$

If I chooses "2",

$$\begin{array}{cc} & \text{III} \\ \text{II} & \begin{pmatrix} (1,0,0) & (1,1,1) \\ (0,0,1) & (0,1,1) \end{pmatrix} \end{array}$$

Aim: to find the characteristic function v.

$v(\phi) = 0$.

$v(N)$ is the largest sum among the eight payoffs, $v(N) = 9$.

To find $v(\{1\})$, we find the payoff matrix for I against (II, III).

$$\begin{array}{cc} & \begin{array}{cccc} (1,1) & (1,2) & (2,1) & (2,2) \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 2 & 4 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

The 2nd col is dominated by 1st col, 3rd col is dominated by 4th col

Thus, we need find the value of the zero-sum game

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We have $v(\{1\}) = 1/2$.

To find $v(\{2\})$, we find the payoff matrix for II against (I, III).

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} (1,1) & (1,2) & (2,1) & (2,2) \end{array} \\ \begin{pmatrix} 3 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Note, the choice of row 2 and column $(2,1)$ is a saddle point.

We have $v(\{2\}) = 0$.

To find $v(\{3\})$, we find the payoff matrix for III against (I, II).

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} (1,1) & (1,2) & (2,1) & (2,2) \end{array} \\ \begin{pmatrix} 1 & 3 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

We have $v(\{3\}) = 3/4$.

To find $v(\{1,3\})$, we find the payoff matrix for (I, III) against II.

$$
\begin{array}{cc}
 & \begin{array}{cc} 1 & 2 \end{array} \\
\begin{array}{c} (1,1) \\ (1,2) \\ (2,1) \\ (2,2) \end{array} & \begin{pmatrix} 1 & 7 \\ 3 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}
\end{array}
$$

The last two rows are dominated by the second row.

Hence $v(\{1,3\}) = 5/2$.

To find $v(\{1,2\})$, we find the payoff matrix for (I, II) against III.

$$\begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} (1,1) \\ (1,2) \\ (2,1) \\ (2,2) \end{array} & \begin{pmatrix} 3 & 3 \\ 6 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \end{array}$$

It has saddle point at row 1 and column 2. Hence $v(\{1,2\}) = 3$.

Similarly, one can find $v(\{2,3\}) = 2$.

Constant-sum games

A game in strategic form is said to be zero-sum if

$$\sum_{i \in N} u_i(x_1, \cdots, x_n) = 0$$

for all strategy choice $(x_1, \cdots, x_n)$.

Thus, for any coalition S, we have

$$\sum_{i \in S} u_i(x_1, \cdots, x_n) = -\sum_{i \in \overline{S}} u_i(x_1, \cdots, x_n)$$

So, we have $v(S) + v(\overline{S}) = 0$.

A game in strategic form is said to be constant-sum if

$$\sum_{i \in N} u_i(x_1, \cdots, x_n) = c$$

for all strategy choice $(x_1, \cdots, x_n)$ and some constant c.

Then, for any coalition S, we have $v(S) + v(\overline{S}) = c = v(N)$.

This leads to the following definition.

Definition: A game in coalitional form is said to be **constant-sum** if $v(S) + v(\overline{S}) = v(N)$ for all coalitions $S \in 2^N$. It is said to be **zero-sum** if $v(N) = 0$.

Imputations

In many-person cooperative games, it is the joint benefit of the players to form the grand coalition N.

The value $v(N)$ is better than the total amount received by other disjoint sets of coalitions they form.

It is then reasonable to assume that the players are "rational" and will agree to form the grand coalition.

Then we need to discuss how the payoff is split among players.

Let $x = (x_1, \cdots, x_n)$ be the payoff vector.

Here, x_i is the amount received by Player i.

Definition: A payoff vector x is said to be **group rational** or **efficient** if $\sum_{i=1}^{n} x_i = v(N)$.

Note that, no player could expect payoff less than $v(\{i\})$.

Definition: A payoff vector x is said to be **individually rational** if $x_i \geq v(\{i\})$.

Definition: An **imputation** is a payoff vector that is group rational and individually rational.

One example of an imputation is $x_i = v(\{i\})$, for $i = 1, \cdots, n-1$, and set $x_n = v(N) - \sum_{i=1}^{n-1} x_i$. (most preferred by Player n)

The set of all imputations is the convex hull of the n points formed in this way.

Example: Consider a game with $v(\{1\}) = 1/2$, $v(\{2\}) = 0$, $v(\{3\}) = 3/4$ and $v(N) = 9$. (ref. previous example)

The set of imputations satisfies

$$x_1 + x_2 + x_3 = 9, \quad x_1 \geq \frac{1}{2}, \quad x_2 \geq 0, \quad x_3 \geq \frac{3}{4}$$

The set of imputations is the triangle with vertices $(33/4, 0, 3/4)$, $(1/2, 31/4, 3/4)$ and $(1/2, 0, 17/2)$.

(imputations most preferred by Players 1, 2 and 3 resp.)

Essential games

When the set of imputations contains only one point, the game is called inessential.

Definition: A game in coalitional form is said to be **inessential** if $\sum_{i=1}^{n} v(\{i\}) = v(N)$, and **essential** if $\sum_{i=1}^{n} v(\{i\}) < v(N)$.

If a game is inessential, only imputation $x = (v(\{1\}), \cdots, v(\{n\}))$. Thus, each player receives his safety level.

Note: two-person zero-sum game is inessential.

In inessential games, there is no tendency for players to form coalitions, as $v(S) = \sum_{i \in S} v(\{i\})$, for all coalitions S.

Example: in the previous example, we have $v(\{1\}) + v(\{2\}) + v(\{3\}) = 1/2 + 0 + 3/4 < 9 = v(N)$. Hence the game is essential.

The Core

Suppose x is a division of $v(N)$.

If there exists a coalition S such that $\sum_{i \in S} x_i < v(S)$, there will be a tendency for members of S to form a coalition (and hence receive more payoffs).

Definition: An imputation x is said to be **unstable through a coalition** S if $v(S) > \sum_{i \in S} x_i$. We say x is **unstable** if there is a coalition S such that x is unstable through S. Otherwise, we say x is **stable**.

Definition: The set C of all stable imputations is called the **core**.

$$C = \{x \mid \sum_{i \in N} x_i = v(N), \quad \sum_{i \in S} x_i \geq v(S), \ \forall S \subset N\}$$

Remark: the core may be empty.

Theorem: The core of an essential constant-sum game is empty.

Proof. Let x be an imputation.

Since the game is essential, we have $\sum_{i \in N} v(\{i\}) < v(N)$.

There is x_k such that $x_k > v(\{k\})$.

(otherwise, $v(N) = \sum_{i \in N} x_i \leq \sum_{i \in N} v(\{i\}) < v(N)$)

Since the game is constant-sum, $v(N - \{k\}) + v(\{k\}) = v(N)$.

Then x is unstable through $S = N - \{k\}$, since

$$\sum_{i \in S} x_i = \sum_{i \in N} x_i - x_k < v(N) - v(\{k\}) = v(N - \{k\}) = v(S)$$

Example: find the core of the game with the following characteristic function

$$v(\phi) = 0 \qquad \begin{array}{l} v(\{1\}) = 1 \\ v(\{2\}) = 0 \\ v(\{3\}) = 1 \end{array} \qquad \begin{array}{l} v(\{1,2\}) = 4 \\ v(\{1,3\}) = 3 \\ v(\{2,3\}) = 5 \end{array} \qquad v(\{1,2,3\}) = 8$$

The set of all imputations x is defined by

$$x_1 + x_2 + x_3 = 8, \quad x_1 \geq 1, \quad x_2 \geq 0, \quad x_3 \geq 1$$

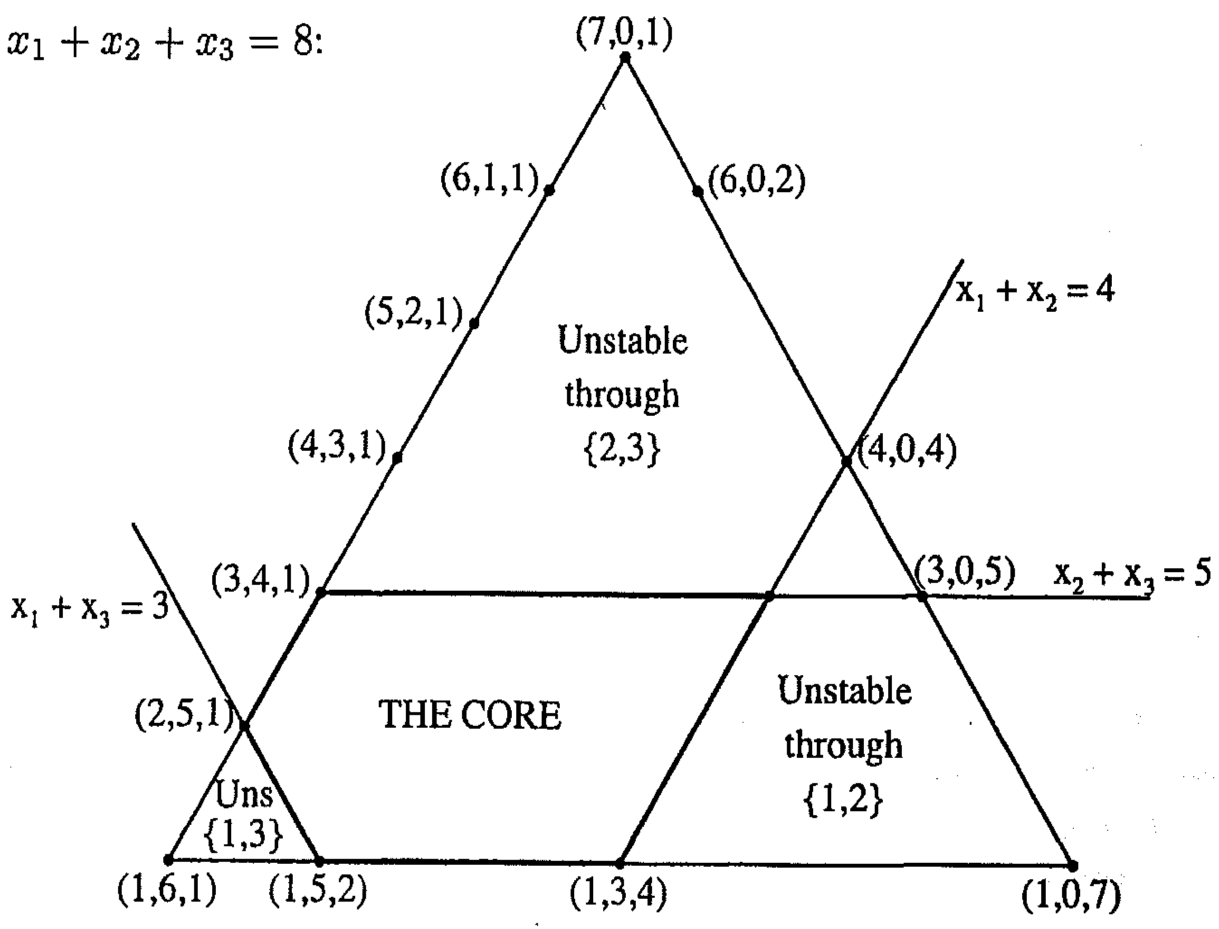
Since $v(\{2,3\}) = 5$, all points with $x_2 + x_3 < 5$ are unstable.

Since $v(\{1,2\}) = 4$, all points with $x_1 + x_2 < 4$ are unstable.

Since $v(\{1,3\}) = 3$, all points with $x_1 + x_3 < 3$ are unstable.

See Figure in next page.

On the plane $x_1 + x_2 + x_3 = 8$:



Example: An object is worth a_i dollars for Player i, $i = 1, 2, 3$.

$(a_1 < a_2 < a_3)$

Player 1 owns the object, and thus $v(\{1\}) = a_1$.

For Players 2 and 3, $v(\{2\}) = 0$, $v(\{3\}) = 0$. Also, $v(\{2, 3\}) = 0$.

Moreover, $v(\{1, 2\}) = a_2$, $v(\{1, 3\}) = a_3$ and $v(\{1, 2, 3\}) = a_3$.

We will find the core:

$$\begin{array}{ccc} x_1 \geq a_1 & x_1 + x_2 \geq a_2 & \\ x_2 \geq 0 & x_1 + x_3 \geq a_3 & x_1 + x_2 + x_3 = a_3 \\ x_3 \geq 0 & x_2 + x_3 \geq 0 & \end{array}$$

Note $x_2 = a_3 - x_1 - x_3 \leq a_3 - a_3 = 0$. Hence $x_2 = 0$.

So, $x_1 \geq a_2$ and $x_3 = a_3 - x_1$.

The core is $\{(x, 0, a_3 - x) \mid a_2 \leq x \leq a_3\}$.

(continue...)

An object is worth a_i dollars for Player i, $i = 1, 2, 3$. $(a_1 < a_2 < a_3)$

The core is $\{(x, 0, a_3 - x) \mid a_2 \leq x \leq a_3\}$.

From this, we see that Player 3 will buy the object with price x $(a_2 \leq x \leq a_3)$.

Player 1 ends up with x dollars.

Player 3 ends up with the object minus x dollars.

Player 2 plays no active role. But without Player 2, Player 3 may get the object for a cheaper price.

The Shapley value

The core gives a set of imputations but does not distinguish one point of the set as preferable to another.

The core may even be empty.

Now, we will introduce the concept of value of a many-person game.

It is a unique payoff vector, such that the i-th component is the payoff to the i-th player.

One can see that the value is a fair distribution of $v(N)$.

Definition: A **value function**, ϕ, is a function that assigns to each characteristic function of an n-person game v to an n-tuple

$$\phi(v) = (\phi_1(v), \phi_2(v), \cdots, \phi_n(v))$$

Here, $\phi_i(v)$ represents the value of Player i.

We use the following axioms to define fairness.

Shapley Axioms for $\phi(v)$:

1. **Efficiency**. $\sum_{i \in N} \phi_i(v) = v(N)$. (group rationality)
2. **Symmetry**. If i and j satisfy $v(S \cup \{i\}) = v(S \cup \{j\})$ for every coalition not containing i and j, then $\phi_i(v) = \phi_j(v)$.

   (if the game is symmetric in Players i and j, they should have the same value)
3. **Dummy Axiom**. If i satisfies $v(S \cup \{i\}) = v(S)$ for every coalition not containing i, then $\phi_i(v) = 0$.

   (if a player neither helps nor harms any coalition, his value is 0)
4. **Additivity**. If u and v are characteristic functions, then $\phi(u + v) = \phi(u) + \phi(v)$.

   (value of two games played at the same time is equal to the sum of the values of the two games played at different times)

Theorem: There exists a unique value function ϕ satisfying the Shapley Axioms.

We divide the proof into a few steps.

For a given nonempty subset $S \subset N$, we define a special characteristic function

$$w_S(T) = \begin{cases} 1 & \text{if } S \subset T \\ 0 & \text{otherwise} \end{cases}$$

where for all $T \subset N$.

Lemma 1: Any characteristic function v can be uniquely written

$$v = \sum_{S \subset N} c_S w_S$$

for some suitable constants c_S.

Proof. Let $c_\phi = 0$. We define c_T inductively on the number of elements in T. Define

$$c_T = v(T) - \sum_{S \subset T, S \neq T} c_S$$

Then we have

$$\sum_{S \subset N} c_S w_S(T) = \sum_{S \subset T} c_S = c_T + \sum_{S \subset T, S \neq T} c_S = v(T)$$

Thus, we have $v = \sum_{S \subset N} c_S w_S$.

Next we prove uniqueness.

Assume there are c_S and c'_S such that

$$v(T) = \sum_{S \subset N} c_S w_S(T) = \sum_{S \subset N} c'_S w_S(T)$$

for all $T \subset N$.

We use induction on the number of elements in T.

Let $T = \{i\}$ be a set of one element. Then all terms above are zero except $S = \{i\}$. Hence we have $c_{\{i\}} = c'_{\{i\}}$.

Let R be an arbitrary set. Assume that $c_S = c'_S$ for all $S \subset R$ and $S \neq R$. Using $T = R$ in the above formula, we have

$$\sum_{S \subset R} c_S w_S(R) = \sum_{S \subset R} c'_S w_S(R)$$

This implies $c_R = c'_R$. The proof is complete.

Note that the unique representation $v = \sum_{S \subset N} c_S w_S$ shows that there is a unique way to find $\phi(v)$.

Assume ϕ exists.

By Axiom 4, we have $\phi(v) = \sum_{S \subset N} \phi(c_S w_S)$.

By Axiom 3, $\phi_i(c_S w_S) = 0$ if $i \notin S$.

By Axiom 2, if both i and j are in S, then $\phi_i(c_S w_S) = \phi_j(c_S w_S)$

By Axiom 1, $\sum_{i \in N} \phi_i(c_S w_S) = c_S w_S(N) = c_S$. This implies that $\phi_i(c_S w_S) = c_S/|S|$ if $i \in S$.

Thus,

$$\phi_i(v) = \sum_{S \subset N} \phi_i(c_S w_S) = \sum_{S \subset N, i \in S} \frac{c_S}{|S|}$$

Example: Find the Shapley value for v:

$$v(\phi) = 0 \qquad \begin{array}{l} v(\{1\}) = 1 \\ v(\{2\}) = 0 \\ v(\{3\}) = 1 \end{array} \qquad \begin{array}{l} v(\{1,2\}) = 4 \\ v(\{1,3\}) = 3 \\ v(\{2,3\}) = 5 \end{array} \qquad v(\{1,2,3\}) = 8$$

We use the inductive formula for c_T.

For sets with one element, $c_{\{1\}} = v(\{1\}) = 1$, $c_{\{2\}} = v(\{2\}) = 0$ and $c_{\{3\}} = v(\{3\}) = 1$.

For sets with two elements, $c_{\{1,2\}} = v(\{1,2\}) - c_{\{1\}} - c_{\{2\}} = 3$,
$c_{\{1,3\}} = v(\{1,3\}) - c_{\{1\}} - c_{\{3\}} = 1$ and
$c_{\{2,3\}} = v(\{2,3\}) - c_{\{2\}} - c_{\{3\}} = 4$.

For the set with 3 elements,
$c_{\{1,2,3\}} = v(\{1,2,3\}) - c_{\{1\}} - c_{\{2\}} - c_{\{3\}} - c_{\{1,2\}} - c_{\{1,3\}} - c_{\{2,3\}} = -2$.

Then we have

$$\phi_1(v) = c_{\{1\}} + \frac{1}{2}c_{\{1,2\}} + \frac{1}{2}c_{\{1,3\}} + \frac{1}{3}c_{\{1,2,3\}} = \frac{14}{6}$$

$$\phi_2(v) = c_{\{2\}} + \frac{1}{2}c_{\{1,2\}} + \frac{1}{2}c_{\{2,3\}} + \frac{1}{3}c_{\{1,2,3\}} = \frac{17}{6}$$

$$\phi_3(v) = c_{\{3\}} + \frac{1}{2}c_{\{1,3\}} + \frac{1}{2}c_{\{2,3\}} + \frac{1}{3}c_{\{1,2,3\}} = \frac{17}{6}$$

Hence the Shapley value is $\phi = (14/6, 17/6, 17/6)$.

Example: A bankruptcy game.

A small company goes bankrupt owing money to three creditors.

The company owns A \$10,000, B \$20,000 and C \$30,000.

The company only has \$36,000 to cover these debts.

How should the money be divided among A, B and C ?

The pro rata split of the money would lead to the allocation of \$6,000 for A, \$12,000 for B and \$18,000 for C.

What would be the allocation if we use the Shapley value?

First, we find a characteristic function.

Note that $v(\phi) = 0$ and $v(ABC) = 36$.

A may get nothing if B and C receive the whole amount, so $v(A) = 0$. Similarly, $v(B) = 0$.

C will get \$6,000 even A and B receive all their claims, so $v(C) = 6$.

A and B will get \$6,000 even C receive all of its claim, so $v(AB) = 6$.

Similarly, we have $v(AC) = 16$ and $v(BC) = 26$.

We need to find the Shapley value for v:

$$
\begin{array}{cccc}
 & v(A) = 0 & v(AB) = 6 & \\
v(\phi) = 0 & v(B) = 0 & v(AC) = 16 & v(ABC) = 36 \\
 & v(C) = 6 & v(BC) = 26 &
\end{array}
$$

We use the inductive formula for c_T.

For sets with one element, $c_{\{A\}} = v(\{A\}) = 0$, $c_{\{B\}} = v(\{B\}) = 0$ and $c_{\{C\}} = v(\{C\}) = 6$.

For sets with two elements, $c_{\{A,B\}} = v(\{A,B\}) - c_{\{A\}} - c_{\{B\}} = 6$, $c_{\{A,C\}} = v(\{A,C\}) - c_{\{A\}} - c_{\{C\}} = 10$ and $c_{\{B,C\}} = v(\{B,C\}) - c_{\{B\}} - c_{\{C\}} = 20$.

For the set with 3 elements, $c_{\{A,B,C\}} =$ $v(\{A,B,C\}) - c_{\{A\}} - c_{\{B\}} - c_{\{C\}} - c_{\{A,B\}} - c_{\{A,C\}} - c_{\{B,C\}} = -6$.

Then we have

$$\phi_A(v) = c_{\{A\}} + \frac{1}{2}c_{\{A,B\}} + \frac{1}{2}c_{\{A,C\}} + \frac{1}{3}c_{\{A,B,C\}} = 6$$

$$\phi_B(v) = c_{\{B\}} + \frac{1}{2}c_{\{A,B\}} + \frac{1}{2}c_{\{B,C\}} + \frac{1}{3}c_{\{A,B,C\}} = 11$$

$$\phi_C(v) = c_{\{C\}} + \frac{1}{2}c_{\{A,C\}} + \frac{1}{2}c_{\{B,C\}} + \frac{1}{3}c_{\{A,B,C\}} = 19$$

Hence the Shapley value is $\phi = (6, 11, 19)$.

Thus, according to the Shapley value, A receives \$6,000, B receives \$11,000 and C receives \$19,000.

Now we discuss the existence of Shapley value.

We first define a value function as follows.

- Suppose we form the grand coalition by entering players one at a time.
- A player receives the amount that he increases the value of the coalition when he enters the coalition.
- This amount depends on the order in which the player enters.
- The value of a player is just the average of these amounts.

We can write this value function as

$$\phi_i(v) = \sum_{S \subset N, i \in S} \frac{(|S|-1)!\,(n-|S|)!}{n!} \left\{ v(S) - v(S - \{i\}) \right\}$$

Lemma 2: The value function

$$\phi_i(v) = \sum_{S \subset N, i \in S} \frac{(|S|-1)!\,(n-|S|)!}{n!} \Big\{ v(S) - v(S - \{i\}) \Big\}$$

is the Shapley value.

Proof. We need to check the 4 axioms.

Axiom 4 holds since the above formula is linear in v.

Axiom 3 holds since $v(S) = v(S - \{i\})$ if $i \in S$.

Axiom 1 holds by the construction.

Simple games

<u>Motivation</u>: In voting games, players vote for a certain bill. The bill is either passed or rejected.

The subsets of players that can pass the bill are called winning coalitions, while those cannot are called losing coalitions.

In this game, we may take the values of winning coalitions to be 1 and losing coalitions to be 0.

This motivates simple games.

Definition: A game (N, v) is **simple** if for every coalition $S \subset N$, either $v(S) = 0$ or $v(S) = 1$.

In simple games, a coalition S is said to be **winning coalition** if $v(S) = 1$, and **losing coalition** if $v(S) = 0$.

Example: $v(S) = 1$ if $|S| > n/2$ otherwise $v(S) = 0$.

Recall the formula for Shapley value:

$$\phi_i(v) = \sum_{S \subset N, i \in S} \frac{(|S|-1)!\,(n-|S|)!}{n!} \Big\{ v(S) - v(S - \{i\}) \Big\}$$

For simple games, it becomes

$$\phi_i(v) = \sum_{i \in S \text{ winning, } S - \{i\} \text{ losing}} \frac{(|S|-1)!\,(n-|S|)!}{n!}$$

This is called the **Shapley-Shubik Power Index**. It measures the power of Player i in the game.

One class of simple games is called **weighted voting games**.

Each player has a **weight** $w_i \geq 0$.

There is a positive number q called the **quota**.

The characteristic function is defined as

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i > q \\ 0 & \text{if } \sum_{i \in S} w_i \leq q \end{cases}$$

If we take $q = \frac{1}{2} \sum_{i \in N} w_i$, this is called a **weighted majority game**.

Example: Player $1, 2, 3$ and 4 have $10, 20, 30$ and 40 shares of stocks respectively. In order to pass certain decision, 50% of the shares are required.

This is a weighted majority game with weights $w_1 = 10$, $w_2 = 20$, $w_3 = 30$ and $w_4 = 40$. And $q = 50$. We find the Shapley-Shubik Power Index.

For $i = 1$, the winning coalitions are $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$ and N. But only $\{1, 2, 3\}$ will be losing when Player 1 quits.

$$\phi_1(v) = \frac{2!1!}{4!} = \frac{1}{12}$$

For $i = 2$, the winning coalitions are $\{2, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{2, 3, 4\}$ and N. But only $\{2, 4\}$, $\{1, 2, 3\}$ and $\{1, 2, 4\}$ are losing without 2.

$$\phi_2(v) = \frac{1!2!}{4!} + \frac{2!1!}{4!} + \frac{2!1!}{4!} = \frac{1}{4}$$

For $i = 3$, the winning coalitions are $\{3,4\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{2,3,4\}$ and N. But only $\{3,4\}$, $\{1,2,3\}$ and $\{1,2,4\}$ are losing without 3.

$$\phi_3(v) = \frac{1!2!}{4!} + \frac{2!1!}{4!} + \frac{2!1!}{4!} = \frac{1}{4}$$

For $i = 4$, the winning coalitions are $\{2,4\}$, $\{3,4\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$, $\{2,3,4\}$ and N. But only $\{2,4\}$, $\{3,4\}$, $\{1,2,4\}$ $\{1,3,4\}$ and $\{2,3,4\}$are losing without 4.

$$\phi_4(v) = \frac{1!2!}{4!} + \frac{1!2!}{4!} + \frac{2!1!}{4!} + \frac{2!1!}{4!} + \frac{2!1!}{4!} = \frac{5}{12}$$

Hence the Shapley-Shubik Power Index is
$\phi = (1/12, 3/12, 3/12, 5/12)$.

Note: Player 2 and 3 have the same power even though Player 3 has more votes.

MAT 4250: Lecture 9

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Chapter 5: Proof of Nash Theorem

Nash Theorem for strategic equilibrium

Consider 2-person noncooperative game with bimatrix (A, B).

Here, A and B are $m \times n$ matrices.

Let

$$\mathcal{P} = \left\{ p = (p_1, \cdots, p_m) \mid p_i \geq 0, \ \sum p_i = 1 \right\}$$

be the set of mixed strategies for Player I and let

$$\mathcal{Q} = \left\{ q = (q_1, \cdots, q_n) \mid p_i \geq 0, \ \sum q_j = 1 \right\}$$

be the set of mixed strategies for Player II.

By definition, $(p^*, q^*) \in \mathcal{P} \times \mathcal{Q}$ is a strategic equilibrium (SE) if

$$p^T A q^* \leq (p^*)^T A q^* \quad \forall p \in \mathcal{P}$$

and

$$(p^*)^T B q \leq (p^*)^T B q^* \quad \forall q \in \mathcal{Q}$$

Theorem: (**Nash**) Every bimatrix game has at least one SE.

Proof. For given p and q, we define

$$c_i = \max(e_i^T Aq - p^T Aq, 0), \quad i = 1, 2, \cdots, m$$

and

$$d_j = \max(p^T Be_j - p^T Bq, 0), \quad j = 1, 2, \cdots, n$$

Let $p' = (p'_1, \cdots, p'_m)$ and $q' = (q'_1, \cdots, q'_n)$ be

$$p'_i = \frac{p_i + c_i}{1 + \sum c_i}, \quad q'_j = \frac{q_j + d_j}{1 + \sum d_j}$$

Define a mapping $T : \mathcal{P} \times \mathcal{Q} \to \mathcal{P} \times \mathcal{Q}$ by $T(p, q) = (p', q')$.

Note T is continuous and $\mathcal{P} \times \mathcal{Q}$ is a bounded closed convex set.

By the Brouwer's fixed point theorem, the mapping T has a fixed point, namely, there is (p^*, q^*) such that $T(p^*, q^*) = (p^*, q^*)$.

It remains to show that a fixed point is a SE.

Suppose it is not true. Then there exists $\tilde{p}$ such that

$$(\tilde{p})^T A q^* > (p^*)^T A q^*$$

OR there exists $\tilde{q}$ such that

$$(p^*)^T B \tilde{q} > (p^*)^T B q^*$$

Consider the first case only.

There is k such that $e_k^T A q^* > (p^*)^T A q^*$.

(or $(\tilde{p})^T A q^* = \sum_{i=1}^m (\tilde{p}_i e_i)^T A q^* \leq \sum_{i=1}^m \tilde{p}_i ((p^*)^T A q^*) = (p^*)^T A q^*$.)

Thus, we have $c_k > 0$. Consequently $\sum_{i=1}^m c_k > 0$.

Note that among those i with $p_i^* > 0$, there is i such that $e_i^T A q^* \leq (p^*)^T A q^*$. Otherwise,

$$(p^*)^T A q^* = (\sum_{i=1}^{m} p_i^* e_i)^T A q^* = (\sum_{p_i^* > 0} p_i^* e_i)^T A q^* > (p^*)^T A q^*$$

For this choice of i, we have $c_i = 0$.

Thus,

$$(p^*)'_i = \frac{p_i^*}{1 + \sum c_k} < p_i^*$$

Hence $p' \neq p$. This is a contradiction.