THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 7

Taylor Theorem Let $f:(a,b)\to\mathbb{R}$ be a function such that the $n+1$-th derivative exists. Let $p_n(x)$ be the Taylor polynomail of degree n of $f(x)$ at $x=c\in(a,b)$. Then for any $x\in(a,b)$, there exists ξ between c and x such that

$$f(x)=p_n(x)+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}.$$

Taylor series Let $f:(a,b)\to\mathbb{R}$ be a function that has derivatives of all order. Then we define the **Taylor series of f centered at** $c\in(a,b)$ to be the power series

$$T(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(c)}{k!}(x-c)^k=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots.$$

Warning: The Taylor series of an infinitely differentiable function at $x=a$ may or may not be equal to the original function at $x\neq a$, even though the Taylor series may converge for some $x\neq a$.

Operations on Taylor series Let f,g be infinitely differentiable functions and $T_f(x)=\sum_{k=0}^{\infty}a_k(x-c)^k, T_y(x)=\sum_{k=0}^{\infty}b_k(x-c)^k$ be their Taylor series at $x=c$ respectively. Let $\lambda\in\mathbb{R}$. Then we have

1. $T_{\lambda f}(x)=\sum_{k=0}^{\infty}\lambda a_k(x-c)^k$
2. $T_{f+g}(x)=\sum_{k=0}^{\infty}(a_k+b_k)(x-c)^k$
3. $T_{fg}(x)=\sum_{k=0}^{\infty}\sum_{i=0}^{k}(a_ib_{k-i})(x-c)^k$

where $T_h(x)$ is the Taylor series of h at $x=c$.

Uniqueness of series If $\displaystyle\sum_{k=0}^{\infty}a_k(x-c)^k$ and $\displaystyle\sum_{k=0}^{\infty}b_k(x-c)^k$ are two series with real coefficient that are convergent and equal on an open non-empty interval I, then $a_k=b_k$ for all $k=0,1,2,\ldots$.

Differentiating a function defined by a power series Suppose the series $S(x)=\displaystyle\sum_{n=0}^{\infty}a_n(x-c)^n$ converges for any x in an open interval $(c-R,c+R)$ for some $R>0$. When we view $S(x)$ as a function on $(c-R,c+R)$, it is differentiable and the value of its derivative at $x\in(c-R,c+R)$ can be given by a covergent series $S'(x)=\displaystyle\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-c)^n$. In particular, if $T(x)$ is the Taylor series of an infinitely differentiable function $f(x)$ that converges on an open interval $(c-R,c+R)$ for some $R>0$, then $T'(x)$ is the Taylor series of $f'(x)$.

Problems that may be demonstrated in class :

Q1. Consider $f(x) = -\ln(1-x)$. Compute its Taylor series at $x = 0$ and show that if $p_4(x)$ is the Taylor polynomial of degree 4 of f at $x = 0$, then

$$|f(0.1) - p_4(0.1)| < 10^{-5}.$$

Hence compute the value of $\ln 0.9$ by hand up to 4 decimal place.

Q2. Given that $\frac{1}{x^2-3x+2} = \frac{1}{x-2} - \frac{1}{x-1}$, find the Taylor series of $\frac{1}{x^2-3x+2}$ at $x = 0$.

Q3. Find the Taylor series of $\frac{1}{1-x}$ at $x = 2$.

Q4. Let f be a differentiable function satisfying $f'(x) = 1 - x + f(x)$ for all $x \in \mathbb{R}$ and $f(0) = 2$. Show that $f(x)$ is infinitely differentiable and find its Taylor series at $x = 0$.

Q5. Let $f(x) = \begin{cases} \frac{\sin x - x}{x^3} & \text{if } x \neq 0 \\ -\frac{1}{6} & \text{if } x = 0 \end{cases}$. Use Taylor theorem to show that $-\frac{1}{6} \leq f(x) \leq -\frac{1}{6} + \frac{x^2}{120}$ for all $x \in \mathbb{R}$.