**Qu.** $\mathbb{R}^n$ is not compact, but it has a nice property close to compact. **What is it?**

**Locally Compact** A topological space $(X, \mathcal{T})$ is locally compact if $\forall\, x \in X$ $\exists$ compact $K$ such that $x \in \underbrace{\mathring{K} \subset K}_{\text{compact neighborhood}}$

**Danger.** The definition is inconsistent with others. Usually, for a topological property ☼D★, $X$ is locally ☼D★ if $\forall\, x \in X$ $\exists$ a local base of ☼D★-nbhds at $x$. That is, $\forall$ nbhd $U$ of $x$ $\exists$ ☼D★-nbhds $V$ such that $x \in V \subset U$

**Fact**

compact $\Rightarrow$ locally compact

$\exists$ compact nbhd base $\Rightarrow$ locally compact

locally compact $\overset{T_2}{\Rightarrow}$ $\exists$ compact nbhd base   **Why?**

**One-point Compactification**

Given a locally compact $T_2$ space $(X, \mathcal{T})$. Then $\exists$ compact $T_2$ space $(X^*, \mathcal{T}^*)$ such that

(i) $X^* \setminus X$ is a singleton

(ii) $\mathcal{T} = \mathcal{T}^*|_X$

(iii) $X$ is noncompact $\Rightarrow$ $\overline{X} = X^*$

$X$ is compact $\Rightarrow$ $X^* \setminus X$ is isolated

Assume $\infty \notin X$, define $X^* = X \cup \{\infty\}$ and

$$\mathcal{J}^* = \mathcal{J} \cup \{ \{\infty\} \cup \underbrace{(X \setminus K)}_{\text{open as } X \text{ is } T_2} : K \subset X \text{ is compact} \}$$

① Verify that $\mathcal{J}^*$ is a topology

Crucial: $\bigcup_{\alpha \in I} (X \setminus K_\alpha) = X \setminus \bigcap_{\alpha \in I} K_\alpha$ both compact

$\bigcap_{j=1}^{n} (X \setminus K_j) = X \setminus \bigcup_{j=1}^{n} K_j$

② $(X^*, \mathcal{J}^*)$ is Hausdorff

The Key step: $x \in X$, $\infty \in X^*$

$x \in U$, $\infty \in \{\infty\} \cup (X \setminus K)$ and

$U \cap (X \setminus K) = \emptyset \iff x \in U \subset K$

③ $(X^*, \mathcal{J}^*)$ is compact

Key idea: If $X^* = \bigcup_{\alpha \in I} U_\alpha \cup \{\infty\} \cup (X \setminus K)$ then

$\{U_\alpha\}$ covers $K$ and has a finite subcover

④ If $X$ is compact, $\{\infty\} \cup (X \setminus X) \in \mathcal{J}^*$

$\therefore$ $\infty \in \{\infty\}$ is isolated

If $\overline{X} \subsetneqq X^*$, then $\overline{X} = X$

$\exists$ nbhd of $\infty$, $\{\infty\} \cup (X \setminus K)$ disjoint from $X$

Only possible $\{\infty\} = \{\infty\} \cup (X \setminus K)$, $K = X$

$\therefore$ $X$ is compact