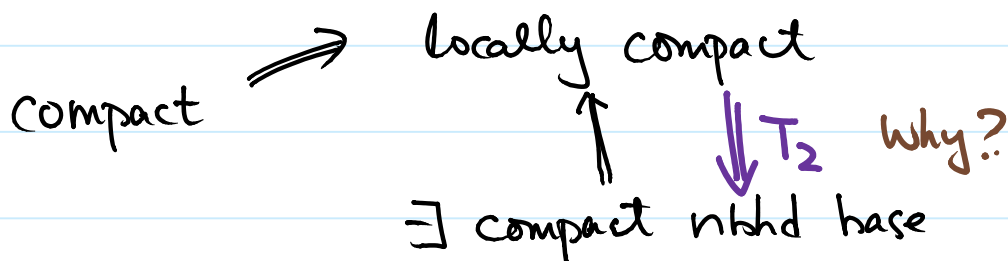


Qu. \mathbb{R}^n is not compact, but it has a nice property close to compact. **What is it?**

Locally Compact A topological space (X, \mathcal{J}) is **locally compact** if $\forall x \in X \exists$ compact K such that $x \in \overset{\circ}{K} \subset K$
compact neighborhood

Danger. The definition is inconsistent with others. Usually, for a topological property \mathcal{P} , X is locally \mathcal{P} if $\forall x \in X \exists$ a local base of \mathcal{P} -nbhds at x . That is, \forall nbhd U of $x \exists \mathcal{P}$ -nbhds V such that $x \in V \subset U$

Fact



One-point Compactification

Given a locally compact T_2 space (X, \mathcal{J})

Then \exists compact T_2 space (X^*, \mathcal{J}^*) such that

(i) $X^* \setminus X$ is a singleton

(ii) $\mathcal{J} = \mathcal{J}^*|_X$

(iii) X is noncompact $\implies \bar{X} = X^*$

X is compact $\implies X^* \setminus X$ is isolated

Assume $\infty \notin X$, define $X^* = X \cup \{\infty\}$ and

$$\mathcal{J}^* = \mathcal{J} \cup \left\{ \{\infty\} \cup \underbrace{(X \setminus K)}_{\text{open as } X \text{ is } T_2} : K \subset X \text{ is compact} \right\}$$

① Verify that \mathcal{J}^* is a topology

Crucial:

$$\bigcup_{\alpha \in I} (X \setminus K_\alpha) = X \setminus \bigcap_{\alpha \in I} K_\alpha$$
$$\bigcap_{j \in J} (X \setminus K_j) = X \setminus \bigcup_{j \in J} K_j$$

both compact

② (X^*, \mathcal{J}^*) is Hausdorff

The key step: $x \in X, \infty \in X^*$

$$x \in U, \infty \in \{\infty\} \cup (X \setminus K) \text{ and}$$

$$U \cap (X \setminus K) = \emptyset \iff x \in U \subset K$$

③ (X^*, \mathcal{J}^*) is compact

Key idea: If $X^* = \bigcup_{\alpha \in I} U_\alpha \cup \{\infty\} \cup (X \setminus K)$ then

$\{U_\alpha\}$ covers K and has a finite subcover

④ If X is compact, $\{\infty\} \cup (X \setminus X) \in \mathcal{J}^*$

$\therefore \infty \in \{\infty\}$ is isolated

If $\bar{X} \subsetneq X^*$, then $\bar{X} = X$

\exists nbhd of ∞ , $\{\infty\} \cup (X \setminus K)$ disjoint from X

Only possible $\{\infty\} = \{\infty\} \cup (X \setminus K)$, $K = X$

$\therefore X$ is compact