Lect19-0329 Cpt Hausdorff

Tuesday, March 29, 2016

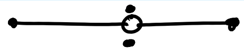
Previously, we proved

$A \subset X$ is closed $\xRightarrow{\text{given } X \text{ is compact}}$ A is compact

Qu. Is it true that

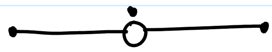
$A \subset X$ is compact $\xRightarrow{X \text{ is compact}}$ A is closed.

Example.

$[-1,1] \sqcup [-1,1] \xrightarrow{q}$ 

compact $\qquad\qquad$ ∴ compact

$[-1,1] \qquad\qquad A = q([-1,1])$



Is it closed?

Theorem Let X be Hausdorff.
If $A \subset X$ is compact then A is closed.

Corollary X is cpt T_2. $A \subset X$ cpt $\Leftrightarrow$ A is closed.

Theorem. Let X be compact, Y be Hausdorff
A continuous bijection $f : X \longrightarrow Y$ is homeomorphic.

Need to show f^{-1} is continuous

$$\begin{array}{ccc}
F \text{ closed in } X & \xleftarrow{f^{-1}} & (f^{-1})^{-1}(F) \text{ closed?} \\
\Downarrow & & \| \qquad \Uparrow \\
F \text{ compact} & \Longrightarrow & f(F) \text{ compact}
\end{array}$$

Let $A \subset X$ be compact and X be Hausdorff

Need to show $A \supset \overline{A}$ or $X \setminus A \in \mathcal{T}$

Take any $x \in X \setminus A$

$\vdots$

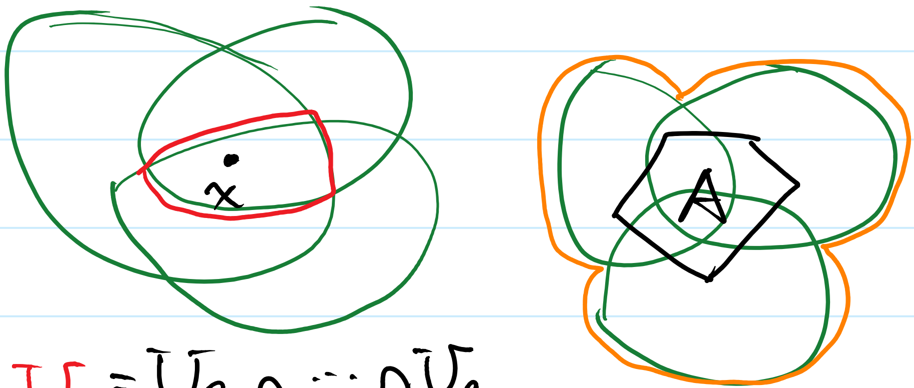
$\exists\, U \in \mathcal{T}$ such that $x \in U \subset X \setminus A$

For each $a \in A$, $x \neq a$

$\exists\, U_a, V_a \in \mathcal{T}$, $x \in U_a$, $a \in V_a$, $U_a \cap V_a = \emptyset$

Then $\mathcal{C} = \{V_a : a \in A\}$ satisfies $\bigcup \mathcal{C} \supset A$

we have $V_{a_1} \cup V_{a_2} \cup \cdots \cup V_{a_n} \supset A$



$U = U_{a_1} \cap \cdots \cap U_{a_n}$

$V = V_{a_1} \cup \cdots \cup V_{a_n}$

Clearly, $x \in U \subset X \setminus V \subset X \setminus A$ $\square$

Actually proved

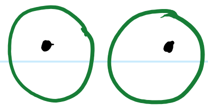
$\forall$ compact $A \subset X$ and $x \notin A$

$\exists\, U, V \in \mathcal{T}$ $x \in U$, $A \subset V$, $U \cap V = \emptyset$

Look familiar?

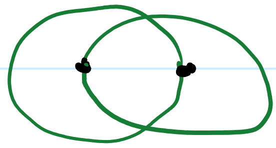
Separation Axioms on $(X, \mathcal{T})$

Hausdorff T_2: $\forall\, x \neq y\ \exists\, U, V \in \mathcal{T}$ such that
$x \in U,\ y \in V,\ U \cap V = \emptyset$



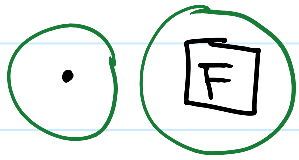
$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$

T_1: $\forall\, x \neq y\ \exists\, U, V \in \mathcal{T}$ such that
$x \in U \setminus V,\ y \in V \setminus U$



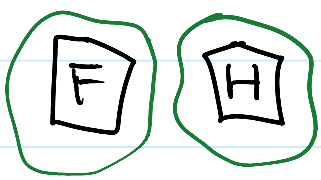
Fact. $T_1 \Leftrightarrow$ singleton is closed
$x \neq y \Leftrightarrow x \in X \setminus \{y\}$
$y \notin U,\ x \in U \Leftrightarrow x \in U \subset X \setminus \{y\}$

Regular: $\forall\, x \notin$ closed F, $\exists\, U, V \in \mathcal{T}$ such that
$x \in U,\ F \subset V,\ U \cap V = \emptyset$



T_3 : $T_1 +$ regular

Normal: $\forall$ closed F, H with $F \cap H = \emptyset$
$\exists\, U, V \in \mathcal{T}$ such that
$F \subset U,\ H \subset V,\ U \cap V = \emptyset$



T_4 : $T_1 +$ normal

Deep Theorem (Urysohn Lemma)
Let $A, B \subset X$ be closed and X be normal.
Then $\exists$ continuous $f : X \longrightarrow [0,1]$ such that
$$f|_A \equiv 0\ ,\quad f|_B \equiv 1\ .$$

Tietz Extension true for normal spaces.

Good about regularity

Let $x \in U$ where $U \in \mathcal{T}$

Then $X \setminus U$ is closed and $x \notin X \setminus U$

By regularity, we have $U_1, V \in \mathcal{T}$ such that

$$\underbrace{x \in U_1, \quad X \setminus U \subset V, \quad U_1 \cap V = \emptyset}$$

$$x \in U_1 \subset X \setminus V \subset U$$

closed

$$\therefore \quad x \in U_1 \subset \overline{U_1} \subset U$$

Thus, we have

$$x \in \cdots\cdots U_n \subset \overline{U_n} \subset U_{n-1} \subset \cdots U_1 \subset \overline{U_1} \subset U$$

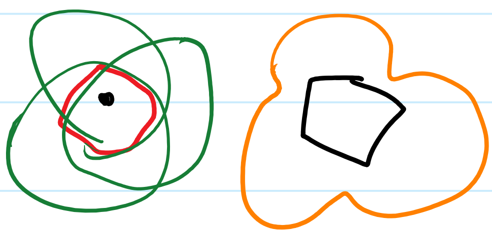
T_3 : T_1 + regular and so T_2

In T_3 space, a closed subset is compact

Then $x \in U_1 \subset K \subset U$, where K is compact

Compact Hausdorff.

From the proof,



X is actually regular, with given T_2, $\therefore$ T_3

Do the proof again for closed F, H, $F \cap H = \emptyset$

X is also normal, $\therefore$ T_4