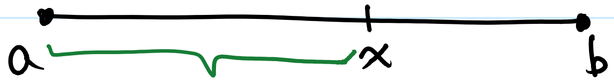
Lect18-0322 Compact

Tuesday, March 22, 2016

Recall $[a,b]$ is compact

Let $\mathcal{G} \subset \mathcal{J}$ be an open cover for $[a,b]$

① 

Ask if $[a,x]$ can have finite subcover

$$T = \left\{ x \in [a,b] : [a,x] \text{ can be covered by finite } \mathcal{F} \subset \mathcal{G} \right\}$$

Then $T \neq \emptyset$ because $a \in T$

Let $s = \sup T$, exists and $s \leq b$

Can prove that $s < b$ gives contradiction

② $[a,b]$ splits into $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ — One of them cannot have finite subcover

Get $[a,b] \supset [a_1, b_1] \supset \cdots \supset [a_k, b_k] \supset \cdots$

If it stops at finite step then done

If it does not stop then contradiction

Note. Second method is valid for closed & bdd subset in $\mathbb{R}^n$, or **totally bounded complete** metric space

Qu. Observe from examples in $\mathbb{R}$, is there any relation between closed & compact?

Theorem. If $(X, \mathcal{J})$ is compact and $A \subset X$ is closed then A is compact

Proof. Let $\mathcal{G} \subset X$ with $\cup\mathcal{G} \supset A$

$\vdots \longleftarrow \mathcal{G}' = \mathcal{G} \cup \{X \setminus A\}, \ \cup\mathcal{G}' = X$

Get finite $\mathcal{E} \subset \mathcal{G}$

$\vdots$

$\cup\mathcal{E} \supset A$

Theorem If $f : (X, \mathcal{J}_X) \longrightarrow Y$ is continuous and X is compact then so is $f(X) \subset Y$.

Proof Let $\mathcal{G} \subset \mathcal{J}_Y$ with $\cup\mathcal{G} \supset f(X)$.

Then $\mathcal{G}_X = \{ f^{-1}V : V \in \mathcal{G} \}$, $\cup\mathcal{G}_X = X$

$\therefore \exists \{ f^{-1}V_1, \cdots, f^{-1}V_n \} \subset \mathcal{G}_X$ satisfies

$$\bigcup_{k=1}^{n} f^{-1} V_k = X$$

Set Theory

$$\bigcup_{k=1}^{n} V_k \supset f(X)$$

* X compact, $X \xrightarrow{\pi} X/\sim \ \Rightarrow X/\sim$ is so.
* $\prod X_\alpha$ compact $\Longrightarrow$ each factor X_β is so.

Qu. What about the converses?

For quotient, e.g., $\mathbb{R} \longrightarrow S^1$.

Theorem If each X_β is compact then

$$\prod_{\alpha\in I} X_\alpha \text{ is also compact}$$

* I is infinite, Tychonoff Theorem
* I is finite, proved below.

Let both X, Y be compact and

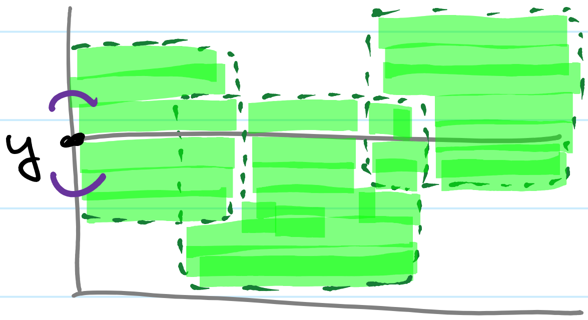
$$\mathcal{C} \subset \mathcal{T}_{X\times Y} \quad \text{with} \quad \bigcup\mathcal{C} = X\times Y$$

For simplicity, assume all sets in $\mathcal{C}$ are of the form $U\times V$ with $U\in\mathcal{T}_X$, $V\in\mathcal{T}_Y$

For each fixed $y\in Y$, $\bigcup\mathcal{C} \supset X\times\{y\}$

$\Sigma_y = \{U_k\times V_k : k=1,\cdots,n\} \subset \mathcal{C}$

$\bigcup_{k=1}^{n} (U_k\times V_k) \supset X\times\{y\}$



$\therefore\ y\in V_k$ for each k

$$y\in V_y = \bigcap_{k=1}^{n} V_k$$

Do this for each y, $\{\bigcup V_y : y\in Y\} \supset Y$

$\exists\ \{V_{y_1}, V_{y_2}, \cdots, V_{y_m}\}$ such that $\bigcup_{l=1}^{m} V_{y_l} \supset Y$

Then, $\bigcup_{l=1}^{m} \Sigma_{y_l} \subset \mathcal{C}$ is a finite subcover for $X\times Y$

For a general $\mathcal{G} \subset \mathcal{T}_{X\times Y}$

finite $\mathcal{E} \subset$

$\mathcal{G}^* \subset \mathcal{T}_{X\times Y}$ of the form $U\times V$

$\mathcal{E}^*$ finite subcover

Qu. Is the following correct?

X is compact $\iff \forall\, \mathcal{G} \subset \mathcal{B}$, a base, with $\bigcup\mathcal{G} = X$, $\exists$ finite $\mathcal{E}\subset\mathcal{G}$, $\bigcup\mathcal{E} = X$

For arbitrary product It is easier to consider intersection of closed sets

1. $\sim(\bigcup\mathcal{G} \supset X) \iff X\setminus\bigcup\mathcal{G} \neq \emptyset$

$$X\setminus\bigcup\mathcal{G} = \bigcap\{\underbrace{X\setminus C}_{\text{closed}} : C\in\mathcal{G}\}, \quad \mathcal{G}\subset\mathcal{T}$$

2. If $\mathcal{G}\subset\mathcal{T}$ with $\bigcup\mathcal{G}=X$ then

$\exists$ finite $\mathcal{E}\subset\mathcal{G}$ with $\bigcup\mathcal{E}=X$

negation $\forall$ finite $\mathcal{E}\subset\mathcal{G}$, $\bigcap\{X\setminus E : E\in\mathcal{E}\}\neq\emptyset$

3. Contrapositive

If then $\bigcap\{X\setminus G : G\in\mathcal{G}\}\neq\emptyset$

X is compact $\Longleftrightarrow$

$\forall$ family $\mathcal{H}$ of closed sets in X

if $\forall$ finite $\mathcal{F} \subset \mathcal{H}$, $\cap \mathcal{F} \neq \phi$ then $\cap \mathcal{H} \neq \phi$

$\Longleftrightarrow$ $\forall$ family $\mathcal{H}$ of sets in X

if $\forall$ finite $\mathcal{F} \subset \mathcal{H}$, $\cap \overline{\mathcal{F}} \neq \phi$ then $\cap \overline{\mathcal{H}} \neq \phi$

$\overline{\mathcal{F}} = \{\overline{F} : F \in \mathcal{F}\}$

finite closure intersection property

Zorn's Lemma

$\forall$ maximal family $\mathcal{M}$ of sets in X with the

f.c.i.p, we have $\cap \overline{\mathcal{M}} \neq \phi$

Good property of maximality of $\mathcal{M}$

(1) It is closed under finite intersection

(2) If $A \subset X$ satisfies $A \cap M \neq \phi$ $\forall M \in \mathcal{M}$

then $A \in \mathcal{M}$

(1) & (2) helps us that

if $x \in \prod X_\alpha$ satisfies $x_\beta \in \overline{\pi_\beta(M)}$ $\forall \beta$ $\forall M \in \mathcal{M}$

then $x \in \overline{M}$ $\forall M \in \mathcal{M}$, $\therefore \cap \overline{\mathcal{M}} \neq \phi$