Lect03-17 Cpt Intro

Wednesday, March 16, 2016 4:48 PM

Qu. How is $[a,b]$ different from (a,b) or $[a,\infty)$?

closed & bdd open unbounded

For the concept of bounded, need metric

i.e., $\exists\ x \in X$, $R > 0$ s.t. $A \subset B(x,R)$

Clearly, expect to remove the metric

Qu. What is the concept?

A. Heine-Borel

B. Bolzano-Weierstrass

C. Sequentially Compact

Given a topological space $(X, \mathcal{T})$.

A set $\mathcal{G} \subset \mathcal{T}$ is an open cover if

$X = \cup\mathcal{G}$ ← the union of all open sets in $\mathcal{G}$

A subset $\mathcal{E} \subset \mathcal{G}$ is a subcover if

it is already an open cover, i.e., $\cup\mathcal{E} = X$

Heine-Borel

The space $(X, \mathcal{T})$ is compact if every open cover has a finite subcover

$\forall\ \mathcal{G} \subset \mathcal{T}$ with $\cup\mathcal{G} = X$, $\exists$ finite $\mathcal{E} \subset \mathcal{G}$

such that $\cup\mathcal{E} = X$.

Let us also recall the other two concepts.

Bolzano-Weierstrass

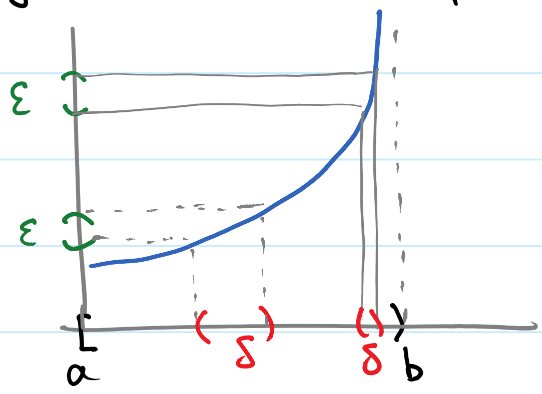
Every infinite set in X has a cluster point.

Sequentially compact

Every sequence has a convergent subsequence.

Each of the three concepts has its importance and usefulness. Let us use the following example to understand **Heine-Borel**.

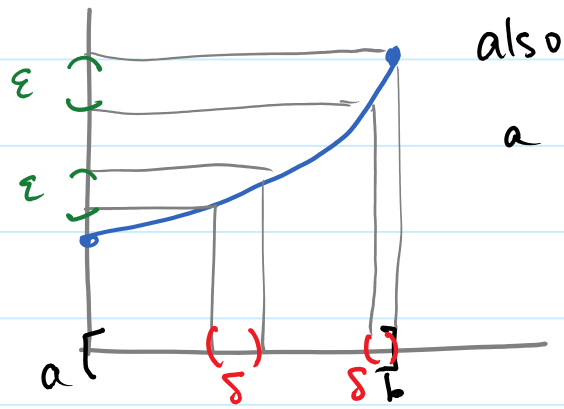
Continuity - The $\delta > 0$ depends on $\varepsilon > 0$ and x



For the same $\varepsilon > 0$, $\delta > 0$ gets smaller and smaller.

There are infinitely many δ-intervals on $[a, b)$

For uniformly continuity, though the δ-intervals



also get smaller. There is a minimum size. Therefore $[a, b]$ can be covered by finitely many δ-intervals

Examples

1. $[a,b]$ is compact. How to prove it?
2. $\mathbb{R}^n$ is not compact
   * $\mathbb{R}^n$ always has a finite open cover, $\mathcal{G}=\{\mathbb{R}^n\}$
     This is irrelevant to compactness
   * $\mathbb{R}^1 = \bigcup_{n\in\mathbb{Z}} (2n, 2n+2) \cup (2n+1, 2n+3)$ but
     cannot be reduced to finitely many.
   * $\mathcal{G} = \{ B(m,1) : m \in \mathbb{Z}\times\mathbb{Z} \}$ is an
     open cover for $\mathbb{R}^2$
     Can we take away some sets from $\mathcal{G}$?
3. $(0,1] = \bigcup_{n\in\mathbb{N}} (\frac{1}{n}, 1]$ is not compact
4. Qu. Is this compact?
   $$K = \{0\} \cup \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$$
   In general, let $x_n \longrightarrow x$ in X. Then
   $$K = \{x\} \cup \{x_n : n \in \mathbb{N}\} \text{ is compact}$$

Compact Subset

Given $(X, \mathcal{T})$ and $A \subset X$.

$(A, \mathcal{T}|_A)$ is compact $\Longleftrightarrow$

$\forall\ \mathcal{G} \subset \mathcal{T}$ with $\bigcup\mathcal{G} \supset A$, $\exists$ finite $\mathcal{E} \subset \mathcal{G}$
such that $\bigcup\mathcal{E} \supset A$