Lect14-0308-Products

Wednesday, March 9, 2016 4:23 PM

Finite Product Given topological spaces $(X_k, \mathcal{T}_k)$ and $P = \prod_{k=1}^{n} X_k$, we consider the set

$$\mathcal{S} = \left\{ X_1 \times X_2 \times \cdots \times X_{k-1} \times U_k \times X_{k+1} \times \cdots \times X_n : k = 1, \cdots, n, \; U_k \in \mathcal{T}_k \right\}$$

This set $\mathcal{S}$ generates the product topology, temporarily denoted as $\mathcal{T}_\Pi$.

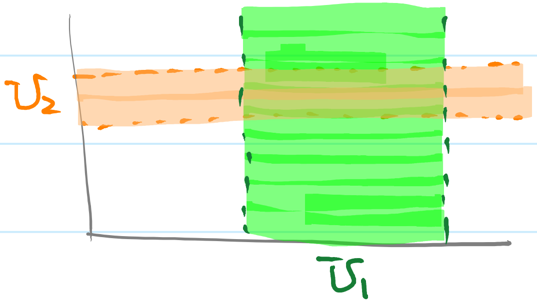
A base for $\mathcal{T}_\Pi$ is

$$\mathcal{B} = \{ U_1 \times U_2 \times \cdots \times U_n : U_k \in \mathcal{T}_k \text{ for } k = 1, \cdots, n \}$$

Question. How to use this as an analogue for infinite product?

Let us study $X_1 \times X_2 \times \cdots \times X_{k-1} \times U_k \times X_{k+1} \times \cdots \times X_n$

A picture for $n=2$ $\qquad \| $ in fact



$\pi_k^{-1}(U_k)$ where

$\pi_k : \prod_{j=1}^{n} X_j \longrightarrow X_k$ is

the projection mapping

In this way, $\mathcal{S} = \{ \pi_k^{-1}(U_k) : k = 1, \cdots, n, \; U_k \in \mathcal{T}_k \}$

We will borrow this notion to infinite product

Question Given sets X_α, $\alpha \in I$, in set language what is $\prod_{\alpha \in I} X_\alpha$? What is $x \in \prod_{\alpha \in I} X_\alpha$?

Actually, x is a function, $x : I \longrightarrow \bigcup_{\alpha \in I} X_\alpha$ such that $x(\alpha) \in X_\alpha$

Example. If $I = \{1, 2, \cdots, n\}$ and $X_k = \mathbb{R}$ for each k

then $x \in \prod_{k=1}^{n} X_k$ is really a function

$$x : \{1, 2, \cdots, n\} \longrightarrow \mathbb{R} = \bigcup_{k=1}^{n} \mathbb{R}$$

and automatically $x(k) \in \mathbb{R}$

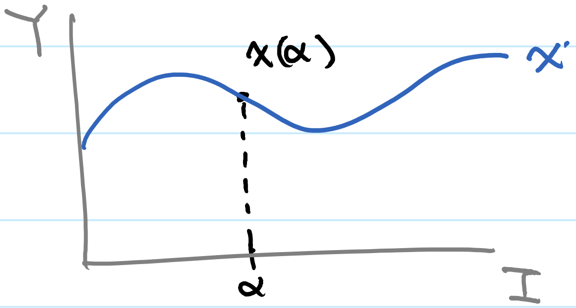
$\| $ write as

x_k

Thus, x is determined by $(x_1, x_2, \cdots, x_n)$.

Example. For general I, and $X_\alpha = Y \ \forall \alpha \in I$

The picture for $x \in \prod_{\alpha \in I} X_\alpha$ can be



In this case, $\prod_{\alpha \in I} X_\alpha = \prod_{\alpha \in I} Y = Y^I$

Infinite Product Topology

Let $(X_\alpha, \mathcal{T}_\alpha)$, $\alpha \in I$ be topological spaces

and $P = \prod_{\alpha \in I} X_\alpha$

The product topology, $\mathcal{T}_\Pi$ g is enerated by

$$\mathcal{S} = \{ \pi_\alpha^{-1}(U_\alpha) : \alpha \in I, \ U_\alpha \in \mathcal{T}_\alpha \}$$

after finite intersection

$\mathcal{B}$, a hase

$\# \; C \ \{ \prod_{\alpha \in I} U_\alpha : U_\alpha \in \mathcal{T}_\alpha \} \rightsquigarrow \mathcal{T}_{box} \neq \mathcal{T}_\Pi$

Example. Let $I = \mathbb{N}$, $X_\alpha = \{0,1\}$ with discrete topology, i.e., $\mathcal{T}_\alpha = \{\phi, \{0\}, \{1\}, \{0,1\}\}$

What is $\{\prod_{\alpha\in I} U_\alpha : U_\alpha \in \mathcal{T}_\alpha\}$?

$$\| $$

$$U_1 \times U_2 \times U_3 \times \cdots \times U_n \times \cdots \times \cdots$$

can be $\phi, \{0\}, \{1\}, \{0,1\}$ $\forall \alpha$

i.e. all possibilities, $\mathcal{T}_{box}$ is discrete

For $\mathcal{S} = \{\pi_\alpha^{-1}(U_\alpha) : \alpha \in I, U_\alpha \in \mathcal{T}_\alpha\}$

$\Big\downarrow$ finite intersection

$$X_1 \times X_2 \times \cdots \times \underbrace{\{0\} \times \cdots \times \{1\}}_{\text{only finitely many singletons}} \times \cdots \times \underbrace{X_n \times X_{n+1} \times \cdots}_{\text{then all will be } \{0,1\}}$$

Question. Why do we take $\mathcal{T}_\Pi$ but not $\mathcal{T}_{box}$?

For general $(X_\alpha, \mathcal{T}_\alpha)$, possible topologies on $P = \prod_{\alpha\in I} X_\alpha$ are

$$\underset{\text{discrete}}{\mathcal{P}(P)} \supset \cdots\cdots \supset \mathcal{T}_{box} \supset \cdots \supset \mathcal{T}_\Pi \supset \cdots \supset \cdots \underset{\text{Indiscrete}}{\{\phi, P\}}$$

Consider each $\pi_\beta : (P, \mathcal{T}) \longrightarrow (X_\beta, \mathcal{T}_\beta)$

To check continuity, take $U_\beta \in \mathcal{T}_\beta$ and verify if $\pi_\beta^{-1}(U_\beta) \in \mathcal{T}$.

True for $\mathcal{T}$ chosen from discrete, $\cdots$, $\mathcal{T}_{box}$, $\cdots$, $\mathcal{T}_\Pi$

But $\mathcal{T}_\Pi$ is the minimal one.

Theorem. The product topology $\mathcal{T}_\Pi$ is the smallest one so that $\forall\ \beta \in I$,

$$\pi_\beta : \prod_{\alpha \in I} X_\alpha \longrightarrow X_\beta \quad \text{is continuous}$$

In other words, if $\pi_\beta : (\prod_{\alpha \in I} X_\alpha, \mathcal{T}) \to X_\beta$ is continuous $\forall\ \beta \in I$, then $\mathcal{T} \supset \mathcal{T}_\Pi$.

This is in fact by construction, $\mathcal{T}_\Pi$ is the smallest containing $\mathcal{S} = \{\pi_\beta^{-1}(U_\beta) : \beta \in I,\ U_\beta \in \mathcal{T}_\beta\}$

Question. Given a function $\mathbb{R}^3 \xrightarrow{f} \mathbb{R}^2$ defined by $(x, y, z) \overset{f}{\longmapsto} (x+y+z, \sin(xyz))$ how do you verify its continuity?

Answer: check $\underbrace{x+y+z}_{\substack{= \\ 1^{st}\ \text{cond.} \\ = \\ \pi_1 \circ f}}$ and $\underbrace{\sin(xyz)}_{\substack{= \\ 2^{nd}\ \text{cond.} \\ = \\ \pi_2 \circ f}}$ separately.

Theorem.

Let $P = \prod_{\alpha \in I} X_\alpha$ be given $\mathcal{T}_\Pi$.

A mapping $f : W \longrightarrow P$ is continuous $\Leftrightarrow \forall\ \beta \in I$, $\pi_\beta \circ f : W \longrightarrow X_\beta$ is continuous

"$\Rightarrow$" is easy; simply because of composition.

"$\Leftarrow$" Let $V \in \mathcal{B}_\Pi$ and we hope to prove $f^{-1}(V) \in \mathcal{T}_W$

Then $V = \pi_{\beta_1}^{-1}(U_1) \cap \pi_{\beta_2}^{-1}(U_2) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_n)$

where $U_k \in \mathcal{T}_{\beta_k}$

Consequently, $f^{-1}(V) = \bigcap_{k=1}^{n} f^{-1}(\pi_{\beta_k}^{-1}(U_k))$

$$= \bigcap_{k=1}^{n} (\pi_{\beta_k} \circ f)^{-1}(U_k)$$

Since each $\pi_{\beta_k} \circ f$ is continuous & $U_k \in \mathcal{J}_{\beta_k}$

$(\pi_{\beta_k} \circ f)^{-1}(U_k) \in \mathcal{J}_W$

and so is V. $\square$

Now, we examine the possible $\mathcal{J}_?$ on $P = \prod_{\alpha \in I} X_\alpha$.
to have the fact:
If each $\pi_\beta \circ f$ is continuous then

$f: W \longrightarrow (P, \mathcal{J}_?)$ is continuous

Obviously, if $\mathcal{J}_? = \{\phi, P\}$, i.e., Indiscrete then f is certainly continuous disregard of $\pi_\beta \circ f$. When more and more sets are put into $\mathcal{J}_?$, we may need to use the continuity of $\pi_\beta \circ f$

Fact. $\mathcal{J}_\Pi$ is actually the maximal topology to have the continuity of f from $\pi_\beta \circ f$.

Exercise. Consider $\prod_{\alpha \in [a,b]} \mathbb{R} = \mathbb{R}^{[a,b]}$, which contains functions $x : [a,b] \longrightarrow \mathbb{R}$. Given a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^{[a,b]}$, what is the meaning of $x_n \longrightarrow x$ in terms of functions?