Lect12-0301-Baire

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Recall Definition Given topological space $(X, \mathcal{T})$

$D \subset X$ is dense if $\overline{D} = X$.

The logical statement is:

$$\underbrace{\forall x \in X \quad \underbrace{x \in \overline{D}}_{\forall U \in \mathcal{T} \text{ with } x \in U,}}_{\forall \phi \neq U \in \mathcal{T}} \quad U \cap D \neq \emptyset$$

Weierstrass Approximation Theorem.

Loosely speaking, any continuous $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ can be approximated by polynomials on "cubes".

$\forall \varepsilon > 0$ $\forall$ interval $[a,b]$, $\exists$ polynomial $p : \mathbb{R}^n \to \mathbb{R}^n$ such that $\sup\{\|p(x) - \varphi(x)\| : x \in [a,b]^n\} < \varepsilon$.

Rephrase in topology

$X = \{\text{continuous functions } \mathbb{R}^n \to \mathbb{R}^n\}$

$P = \{\text{polynomials } \mathbb{R}^n \to \mathbb{R}^n\}$

$\mathcal{T}$ = compact-open topology

Given any $\varphi \in X$, any $U \in \mathcal{T}$ with $\varphi \in U$

$$\underbrace{\exists\, p \in P \text{ such that } p \in U.}_{p \in U \cap P \neq \emptyset}$$

Equivalently, $\overline{P} = X$, i.e., P is dense in X.

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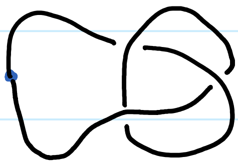
Approximation of knot

A real-life knot is something like

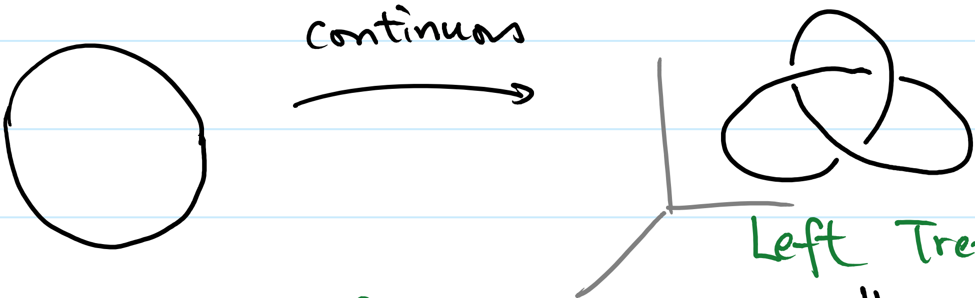
. But it may untie itself under continuous movement

So, we glue up the loose ends to make



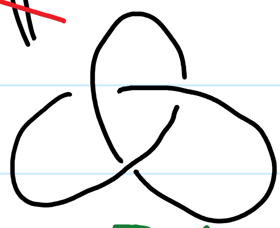
. Then under continuous motion, the knot will still be there.

Mathematically, it is a mapping



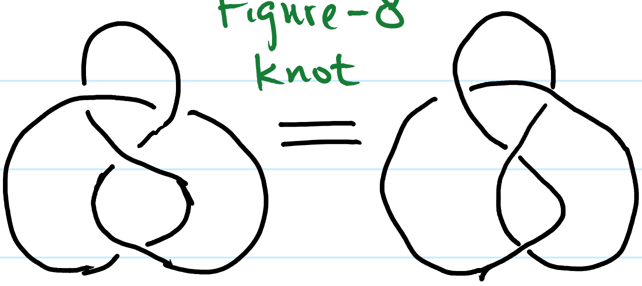
Left Trefoil

$\neq$

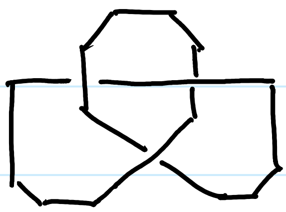


Right Trefoil

Figure-8 knot



Statement. Any knot can be approximated by a polygonal knot



polygonal knot

Again,

$\left\{ \text{polygonal knots} \right\}$ is dense

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Typical Examples

Dense sets in $\mathbb{R}$: $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$

Question. What should be the "opposite" of dense set?

Think about in $\mathbb{R}$, we may say $\mathbb{Z}$.

Question. How to characterize $\mathbb{Z}$?

(a) Closure: $\overline{\mathbb{Z}} = \mathbb{Z}$. Doesn't work, $\overline{[a,b]} = [a,b]$.

(b) Interior: $\mathring{\mathbb{Z}} = \phi$ Doesn't work, $\mathring{\mathbb{Q}} = \phi$ also.

(c) Combining closure and interior

$$(\overline{\mathbb{Z}})^\circ = \mathring{\mathbb{Z}} = \phi \quad \text{seems to work.}$$

Definition. $N \subset X$ is nowhere dense if $(\overline{N})^\circ = \phi$.

Examples

(1) $\mathbb{Z}^n$ is nowhere dense in $\mathbb{R}^n$

(2) Given differentiable functions $x(t), y(t)$

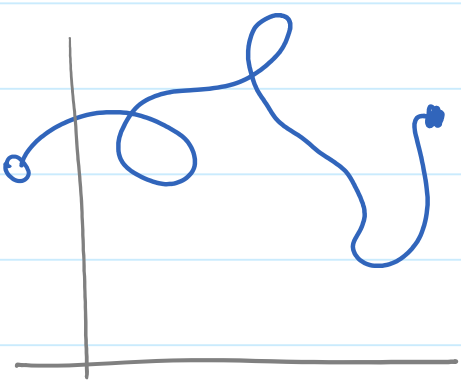
The subset

$$C = \{ (x(t), y(t)) : t \in \text{parameter interval} \}$$

is nowhere dense in $\mathbb{R}^2$.

Note that differentiability is important (the proof needs Inverse Function Thm).

Continuous curve may be "space-filling".



Nowhere Dense

Let us consider the logical statement of

$$(\bar{N})^\circ = \phi$$

$$\forall x \in X, \ \underbrace{x \notin (\bar{N})^\circ}_{\text{negation of } x \in (\bar{N})^\circ}$$

$\sim(\exists\ U \in \mathcal{T}$ with $x \in U,\ U \subset \bar{N})$

i.e. $\forall\ U \in \mathcal{T}$ with $x \in U,\ U \not\subset \bar{N}$

$$\forall\ \phi \neq U \in \mathcal{T}, \ \underbrace{U \not\subset \bar{N}}_{U \cap (X \setminus \bar{N}) \neq \phi}$$

What does this mean?

Recall the meaning of a dense set $\Big\}$ $X \setminus \bar{N}$ is dense

Fact. N is nowhere dense $\Longleftrightarrow$ $X \setminus \bar{N}$ is dense

$\Updownarrow$

$$\underbrace{\forall\ \phi \neq U \in \mathcal{T},\ U \setminus \bar{N} \neq \phi}_{\text{will use it later.}}$$

Question. What is the topological difference between $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$?

Countable and zero measure are not topological

Definition $A \subset X$ is of first category, cat-I if $A = \bigcup_{k=1}^{\infty} N_k$, where N_k are nowhere dense

Otherwise, it is of second category, cat-II

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Examples.

① $\mathbb{Q}$ is of cat-I, $\mathbb{Q} = \bigcup_{k=1}^{\infty} N_k$,

each N_k is a singleton.

Warning. Any countable subset of a space is of cat-I ✗

Although, it can be a countable union of singletons, but $(\overline{\{x\}})^\circ \neq \phi$

② A countable union of cat-I is still cat-I.

③ Question. How do you know $\mathbb{R} \setminus \mathbb{Q}$ is of cat-II?

Answer. Because $\mathbb{R}$ is cat-II and by ② above.

Question. Why is $\mathbb{R}$ of cat-II ?

Answer. Due to the theorem below.

Baire Category Theorem Any complete metric space is of second category.

Proof by contradiction: Assume

$$X = \bigcup_{k=1}^{\infty} N_k \quad \text{where} \quad (\overline{N_k})^\circ = \phi \quad \forall k$$

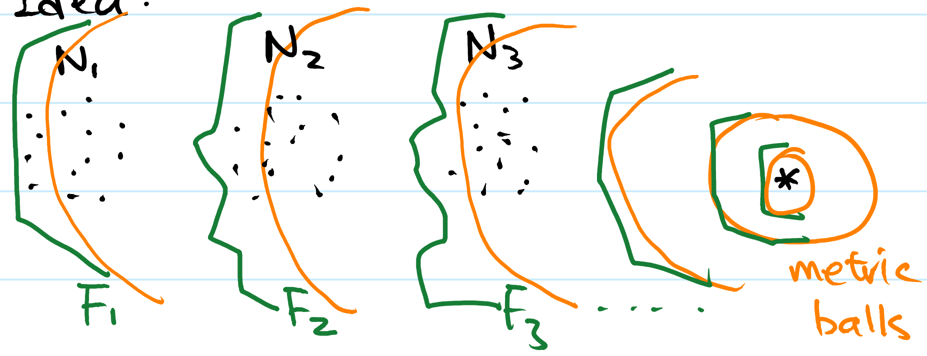
The only information about X is complete, so we use Cauchy sequence or related properties.

Cantor Intersection

Contraction mapping Not here

Idea: construct closed sets F_n, which contain fewer and fewer N_k.

Pictorial Idea:



Now, recall that for a nowhere dense set N,

$\forall\ \phi \neq U \in \mathcal{T}$, $U \setminus \overline{N} \neq \phi$

We will use this repeatedly.

First, $\phi \neq X \in \mathcal{T}$, so $X \setminus \overline{N_1} \neq \phi$, $\therefore \exists\ x_1 \in X \setminus \overline{N_1}$

As $X \setminus \overline{N_1}$ is open, $\exists\ r_1 > 0$ such that

$$x_1 \in B(x_1, 2r_1) \subset X \setminus \overline{N_1}$$

Take $F_1 = \{x \in X : d(x, x_1) \leq r_1\} \subset B(x_1, 2r_1)$

Second, $\phi \neq B(x_1, r_1) \in \mathcal{T}$, so $B(x_1, r_1) \setminus \overline{N_2} \neq \phi$

Similarly, we have $x_2 \in B(x_2, 2r_2) \subset B(x_1, r_1) \setminus \overline{N_2}$

and $F_2 = \{x \in X : d(x, x_2) \leq r_2\} \subset B(x_2, 2r_2)$

Iteratively, we have $x_1, x_2, \cdots, x_n, \cdots \in X$ and

closed sets $F_n \subset B(x_n, 2r_n) \subset X \setminus \overline{N_1} \setminus \overline{N_2} \setminus \cdots \setminus \overline{N_n}$

and $F_{n+1} \subset F_n$ and

$$\operatorname{diam}(F_n) \leq \frac{2r_1}{2^{n-1}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

Thus, $\{x_*\} = \bigcap_{n=1}^{\infty} F_n \subset X \setminus \left(\bigcup_{k=1}^{\infty} \overline{N_k} \right) = \phi$

$\{x_*\} \subset \phi$ is a contradiction