Known Before If $f, g : X \longrightarrow$ Hausdorff are continuous ; $A \subset X$ with $\bar{A} = X$ such that $f|_A \equiv g|_A$. then $f \equiv g$ on X

This is a uniqueness statement, need to known that f, g are already continuous on X.

Theorem Let (X, d_X) and (Y, d_Y) be metric spaces and Y be $\boxed{\text{complete.}}$ If $\bar{A} = X$ $f : A \longrightarrow Y$ is uniformly continuous then $\exists$ unique continuous $\hat{f} : X \longrightarrow Y$ such that $\hat{f}|_A = f$

* Both X, Y have to be metric
* f only initially defined on A, not X
* Need a "stronger" Continuity (only for metric)
* $\hat{f}$ is also "stronger" continuous
* Uniqueness comes from previous theorem

Uniform Continuous $f : (X, d_X) \longrightarrow (Y, d_Y)$ is uniformly continuous if $\forall \ \varepsilon > 0$ $\exists \ \delta > 0$ (only depends on ε) such that

$$\underbrace{\text{if } \ d(x_1, x_2) < \delta \ \text{ then } \ d(f(x_1), f(x_2)) < \varepsilon}$$

$$\forall \ x \in X \quad f(B_X(x, \delta)) \subset B_Y(f(x), \varepsilon)$$

Try to define $\hat{f}(x)$ for $x \in X = \overline{A}$

For each $x \in X$, $\exists$ sequence in A,

call it $a_n^x \longrightarrow x$ as $n \to \infty$

If $x \in A$, choose $a_n^x = x \;\forall n$

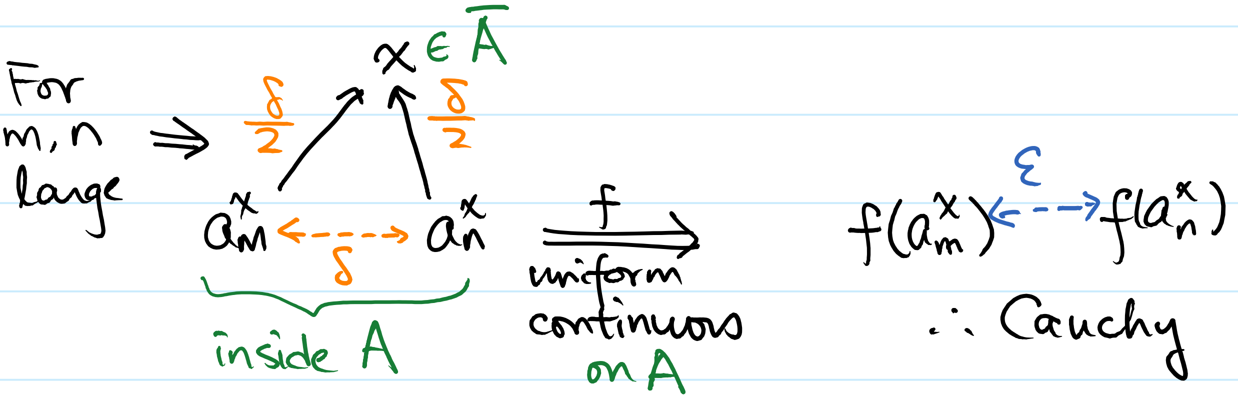
Temporarily, $\hat{f}(x)$ depends on the choice of a_n^x

Then $f(a_n^x)$ is a sequence in Y

Hope. It is Cauchy, $\therefore$ call its limit $\hat{f}(x)$

Take any $\varepsilon > 0$, want to find $N \in \mathbb{N}$ such that

$$\forall\, m,n \geq N \quad d_Y(f(a_m^x), f(a_n^x)) < \varepsilon$$



For the given $\varepsilon > 0$ above, by unif. continuity of f

$\exists\, \delta > 0$ such that if $a, a' \in A$ with

$d_X(a, a') < \delta$ then $d_Y(f(a), f(a')) < \varepsilon$

For such $\delta > 0$, as (a_n^x) converges to x,

$\exists\, N \in \mathbb{N}$ such that if $m, n \geq N$

$$d_X(a_m^x, a_n^x) \leq d_X(a_m^x, x) + d_X(a_n^x, x) < \delta$$

Thus, we have $N \in \mathbb{N}$, if $m, n \geq N$

$$d_Y(f(a_m^x), f(a_n^x)) < \varepsilon$$

The sequence $(f(a_n^x))_{n=1}^\infty$ in Y is Cauchy and has a limit, to be defined as $\hat{f}(x)$

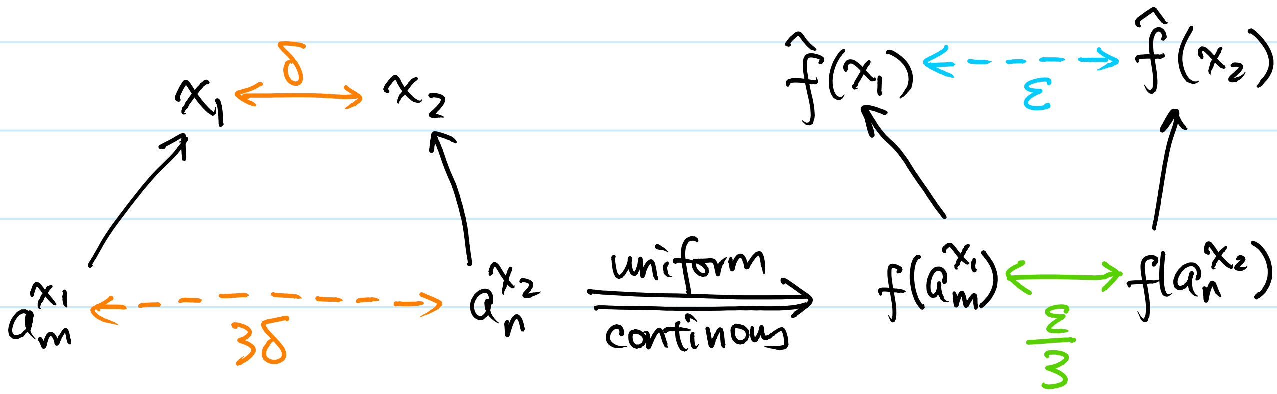
Note that $\hat{f}|_A \equiv f$ by choice of sequence.

If $\hat{f}$ is continuous on X, then by previous result, it is unique, and so indep. of choices of $(a_n^x)_{n=1}^\infty$

Continuity of $\hat{f}$ (uniformly)

Want: $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that

if $d_X(x_1, x_2) < \delta$ then $d_Y(\hat{f}(x_1), \hat{f}(x_2)) < \varepsilon$



Exercise. Write down the ε-δ argument using the idea of the diagram.