Lect10-0218-23-Complete

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Cauchy sequence, a concept only defined on a metric space (X, d). A sequence $(x_n)_{n=1}^{\infty}$ is Cauchy if $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ such that $\forall\ m, n \geq N$ $d(x_m, x_n) < \varepsilon$.

Fact. Every convergent sequence is Cauchy.
The proof requires Δ-inequality.
The converse may not be true, that is, $\exists$ metric space in which a Cauchy sequence may not converge in the space.

Example. $\mathbb{R}^n \setminus \{\bar{0}\}$, then any distinct sequence in $\mathbb{R}^n$ converging to $\bar{0}$ is a Cauchy sequence in $\mathbb{R}^n \setminus \{\bar{0}\}$ but obviously its limit $\bar{0} \notin \mathbb{R}^n \setminus \{\bar{0}\}$

It is clear that such a "bad" can be formed by taking away limits of sequences.

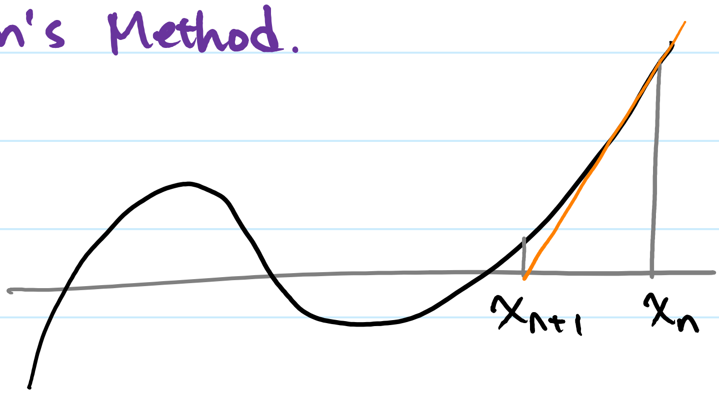
Definition. A metric space is complete if every Cauchy sequence converges in it

Example. Think about how $\mathbb{R}$ is constructed from $\mathbb{Q}$. One method is by equivalence classes of Cauchy sequences in $\mathbb{Q}$. This really is adding limits of Cauchy sequences to $\mathbb{Q}$ and form $\mathbb{R}$.

Question. Why is Cauchy sequence important?

Let us think about when it was used.

Newton's Method.



Pick any x_1 and obtain $x_2, x_3, x_4, \cdots$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If f' is good enough, the sequence (x_n) converges and the limit is a root of f.

Actually, the convergence is guaranteed by that (x_n) is Cauchy.

Inverse / Implicit we know that

$$\text{Analysis} \xrightarrow[\text{Need Cauchy sequence}]{} \text{Inverse Function Thm} \overset{\text{Exercise}}{\Longleftrightarrow} \text{Implicit Function Thm}$$

Example.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\text{given } f} & \mathbb{R} \\ \cup & & \cup \\ U & \xleftarrow{\text{find } f^{-1}} & V \end{array}$$

Main Idea. To define f^{-1} on V,

given $y \in V$, we need $x \in U$, $f(x) = y$

Same as solving $F(x) = f(x) - y = 0$

The proof involves technique similar to Newton's method.

Qu. Given a complete metric space X and $Y \subset X$.

We know that Y may not be complete.

Clearly, $\mathbb{R}^n \setminus \{\vec{0}\} \subset \mathbb{R}^n$ is an example.

Any condition on Y will be enough?

Proposition. Given complete metric space X.

$Y \subset X$ is complete $\Leftrightarrow$ Y is closed.

Note that on a metric space, it is 1^{st} countable.

So $x \in \overline{Y} \Leftrightarrow \exists$ sequence (y_n) in Y such that $y_n \to x$.

"$\Leftarrow$" Given a Cauchy sequence (y_n) in Y, then

it is Cauchy in X (same metric)

So, $\exists\, x \in X$ $y_n \to x$ (X is complete)

Such convergence implies $x \in \overline{Y}$

Thus $x \in Y$ (Y is closed, i.e., $Y = \overline{Y}$)

"$\Rightarrow$" Let $x \in \overline{Y}$, $\exists\, y_n \in Y$, $y_n \to x$ in X (metric)

(y_n) is Cauchy in X, so in Y

$\exists\, y \in Y$ where $y_n \to y$ (Y is complete)

By uniqueness, $y = x$ (X is Hausdorff) $\square$

Remark. Cauchy sequence has been used in analysis besides Newton's method and Inverse Function Theorem.

Qu. How do you prove Intermediate Value Thm?

You may have used a technique of dividing the interval into a half repeated and then apply the Nested Interval Theorem.

Diameter On a metric space (X, d), $A \subset X$

$$\text{diam}(A) = \sup\{ d(a_1, a_2) : a_1, a_2 \in A \}$$

Cantor Intersection Theorem Let (X, d) be a complete metric space; $\phi \neq F_n \subset X$;

* each F_n is closed,
* $F_{n+1} \subset F_n$, i.e., nested
* $\text{diam}(F_n) \longrightarrow 0$ as $n \to \infty$

Then $\bigcap_{n=1}^{\infty} F_n$ is a $\boxed{\text{singleton}}$ Existence Uniqueness

We need to find an element $x \in \bigcap_{n=1}^{\infty} F_n$, i.e., $x \in F_n \ \forall\, n = 1, 2, \cdots$. Naturally, hope to get it by Cauchy Sequence.

Proof. Just pick $x_n \in F_n$ for each n

Then for each $n \in \mathbb{Z}$ and $1 \le p \in \mathbb{Z}$,

$$x_{n+p} \in F_{n+p} \subset F_n \quad \text{and} \quad x_n \in F_n$$

$$\therefore \quad d(x_n, x_{n+p}) \le \operatorname{diam}(F_n) \longrightarrow 0 \quad \text{as } n \to \infty$$

The sequence (x_n) is Cauchy.

$\exists\, x \in X$ such that $x_n \longrightarrow x$ (X is complete)

Why $x \in F_n\ \forall n$?

The sequence $(x_n, x_{n+1}, x_{n+2}, \cdots, \cdots)$ in F_n also converges to x, by definition

$\therefore \quad x \in \overline{F_n} = F_n$ (it is closed)

Finally, $\operatorname{diam}(F_n) \longrightarrow 0$, so there cannot be two distinct element in $\bigcap_{n=1}^{\infty} F_n$.

Thus, $\bigcap_{n=1}^{\infty} F_n = \{x\}$. $\square$

Remark. Cauchy sequences are often indirectly used in analysis.

Contraction Mapping. A mapping $\varphi: X \longrightarrow X$ is a contraction mapping if $\exists$ constant $0 < \alpha < 1$ such that for all $x_1, x_2 \in X$

$$d(\varphi(x_1), \varphi(x_2)) < d(x_1, x_2)$$

Banach Fixed Point Theorem A contraction mapping on a complete metric space always has a fixed point, i.e., $x_0 \in X$ with $\varphi(x_0) = x_0$.

Examples.

In Newton's Method, to solve $f(x)=0$, we have actually created $\varphi: \mathbb{R} \longrightarrow \mathbb{R}$ where

$$\varphi(x) = x - \frac{f(x)}{f'(x)}$$

It is a contraction mapping if f satisfies certain reasonable conditions.

Similarly, to solve for a solution $y(t)$ of an ODE, we create $\varphi: \mathcal{Y} \longrightarrow \mathcal{Y}$ where $\mathcal{Y}$ is a function space; and φ is a contraction mapping such that its fixed point is a solution of the ODE.

Proof. Pick $x_1 \in X$, let $x_{n+1} = \varphi(x_n)$

$$d(x_{n+p}, x_n) \leq d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \cdots + \cdots + d(x_2, x_1)$$

Note that $d(x_m, x_{m-1}) = d(\varphi(x_{m-1}), \varphi(x_{m-2}))$

$$< \alpha \cdot d(x_{m-1}, x_{m-2}) < \alpha^{m-2} d(x_2, x_1)$$

Thus, $d(x_{n+p}, x_n) < \alpha^{n+p-2} d(x_2, x_1) + \cdots + d(x_2, x_1)$

$$= (\alpha^{n+p-2} + \alpha^{n+p-3} + \cdots + \alpha + 1)\, d(x_2, x_1)$$

$$< \alpha^{n+p-1} \cdot d(x_2, x_1) \longrightarrow 0 \text{ as } \alpha < 1$$

$\therefore$ (x_n) is Cauchy, $x_n \longrightarrow x \in Y \subset X$

To finish the proof, we need

As $x_n \to x$, $\varphi(x_n) \to \varphi(x)$

This requires continuity of φ.

Exercise. Every contraction mapping is continuous.

Thus, $$\begin{array}{ccc} \varphi(x_n) & \longrightarrow & \varphi(x) \\ \| & & \\ x_{n+1} & \longrightarrow & x \end{array}$$

By uniqueness, $\varphi(x) = x$. $\square$