Lect05-0126-Countability

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Two ways of describing a base $\mathcal{B}$ of $(X, \mathcal{T})$

* Any $G \in \mathcal{T}$ can be expressed as

$$G = \bigcup_{\alpha \in I} B_\alpha \quad \text{where } B_\alpha \in \mathcal{B}$$

* $\forall\, G \in \mathcal{T}$ and $x \in G$ $\exists\, B \in \mathcal{B}$ such that

$$x \in B \subset G$$

A local base $\mathcal{U}_x$ at the point $x \in X$ satisfies
$\forall$ neighborhood N of x (i.e., $x \in \mathring{N}$)
$\exists\, U \in \mathcal{U}_x$ such that $x \in \mathring{U} \subset U \subset N$

Note Some may require $N, U \in \mathcal{T}$

Both involve inserting an open set between a point $x \in X$ and an arbitrary set.

Example in $(\mathbb{R}^n, \mathcal{T}_{std})$.

$\mathcal{B}_1 = \{B(x,r) : x \in \mathbb{R}^n, r > 0\}$ is a base

It contains many many sets

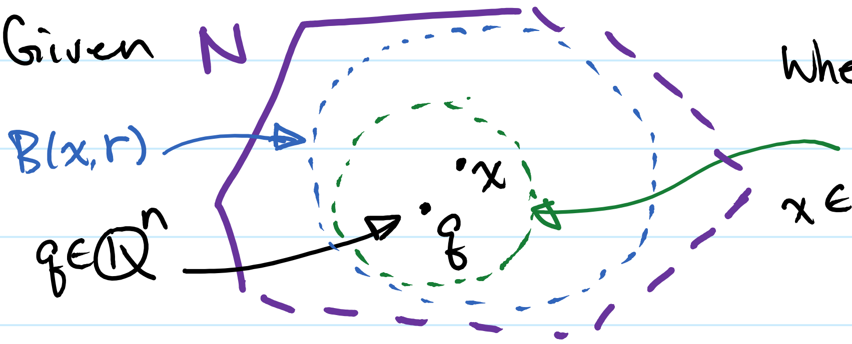
$$= \bigcup_{x \in \mathbb{R}^n} \underbrace{\{B(x,r) : r > 0\}}_{\text{A local base } \mathcal{U}_x \text{ at a fixed } x \in X}$$

$\mathcal{B}_2 = \{B(q, \frac{1}{n}) : q \in \mathbb{Q}^n, 1 \le n \in \mathbb{Z}\}$ is also a base; it is countable

Qu. For $\mathcal{B}_2$, what is the corresponding local base for a fixed $x \in \mathbb{R}^n \setminus \mathbb{Q}^n$?

Now, one cannot use x as the center!

Given N



When $\|q-x\|<\delta$

$x \in B(q, \frac{1}{n}) \subset N$

The argument needs

* $\exists$ sequence $q_k \longrightarrow x$ in $\mathbb{R}^n$
* Δ-inequality $B(q, \frac{1}{n}) \subset B(x, r)$

Qu. What is the relation between a base $\mathcal{B}$ and a local base $\mathcal{U}_x$ at a point $x \in X$?

* $\mathcal{B}$ is indeed a local base for each $x \in X$.
* If we have local base $\mathcal{U}_x$ at every $x \in X$, then $\mathcal{B} = \bigcup_{x \in X} \mathcal{U}_x$ is a base

Definition A topological space $(X, \mathcal{J})$ is

(i) 2^{nd} countable C_{II} if it has a countable base

(ii) 1^{st} countable C_I if there is a countable local base at every $x \in X$.

(iii) Will define later.

Fact: Obvious, $G_{II} \Rightarrow G_I$ ~~$\Leftarrow$~~ should expect

As mention, if we have local base $\mathcal{U}_x$

$$\mathcal{B} = \underbrace{\bigcup_{x \in X}}_{\text{may be uncountable}} \mathcal{U}_x \text{ is a base}$$

In $(\mathbb{R}^n, \mathcal{T}_{std})$, we may use $\underbrace{q \in \mathbb{Q}^n}_{\text{countable}}$ instead and every $x \in \mathbb{R}^n$ can be approximated by $q_k \in \mathbb{Q}^n \longrightarrow x$

Definition. A set $D \subset X$ is dense if $\overline{D} = X$

Qu Recall the logical statement for $x \in \overline{D}$

i.e. $\forall\, U \in \mathcal{T}$ with $x \in U$, $\underbrace{U \cap D \neq \phi}_{\exists\, d \in D \text{ and } d \in U}$

If $\overline{D} = X$ then becomes for all $x \in X$

Thus $\overline{D} = X \Leftrightarrow \forall\, \phi \neq G \in \mathcal{T}, \quad G \cap D \neq \phi$

Definition (iii) A topological space $(X, \mathcal{T})$ is separable if it has a countable dense set.

The obvious example is $\mathbb{R}$ has $\mathbb{Q}$.

Expectation. $G_I \not\Rightarrow$ separable

occurs at $x \in X$

a condition over the whole X

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Theorem. $C_{II} \Rightarrow$ separable

Let $\mathcal{B} = \{B_j : j \in \mathbb{N}\}$ be a countable base

we need to construct a countable dense D

No other idea, so just pick $x_j \in B_j$ and form $D = \{x_j : j \in \mathbb{N}\}$

Let $G \in \mathcal{J}$ with $G \neq \emptyset$, so G is a union of sets in $\mathcal{B}$, namely, $G = \bigcup_k B_{j_k}$

Then clearly, $B_{j_k} \in G \cap D$.

From this fact, we see that a base $\mathcal{B}$ indeed contains a lot of sets. Just take one point from each will form a dense set.

Apparently, if we have a dense set D and each point has a local base $\mathcal{U}_x$, then one may try to make $\bigcup_{d \in D} \mathcal{U}_d$.

Expect. C_I + separable $\Rightarrow C_{II}$

✗ not true however.

Qu. What makes the case of $\mathbb{R}^n$ work?

(i) $\mathbb{Q}^n$, separable

(ii) metric $\to$ C_I, Δ-inequality

Example. Lower Limit topology on $\mathbb{R}$

$\mathcal{T}$ is generated by $\{\phi\} \cup \{ [a,b) : a < b \in \mathbb{R} \}$

* It is obviously C_I. Why?

At every $x \in \mathbb{R}$, take

$$\mathcal{U}_x = \{ [x, x+\tfrac{1}{n}) : 1 \leq n \in \mathbb{Z} \}$$

* It is also separable. Explain why $\mathbb{Q}$ is a dense subset in this topology.

* $\mathcal{C} = \{\phi, \mathbb{R}\} \cup \{ [p,q) : p < q \in \mathbb{Q} \}$ is not a base. With some similar argument, one may rigorously show that this topology is not C_{II}.

Just consider $x \in \mathbb{R} \setminus \mathbb{Q}$ and $[x, x+\varepsilon)$.

Can we insert a subset $[p,q)$, i.e.,

$$x \in [p,q) \subset [x, x+\varepsilon)$$

Obviously, due to $x \notin \mathbb{Q}$ and $p \in \mathbb{Q}$,

$$x \in [p,q) \Rightarrow x \lneq p$$

$$[p,q) \subset [x, x+\varepsilon) \Rightarrow p \lneq x$$

Continuity of a mapping

$$f:(X,\mathcal{T}_X) \longrightarrow (Y,\mathcal{T}_Y)$$

is a property that respects / preserves the topological structure.

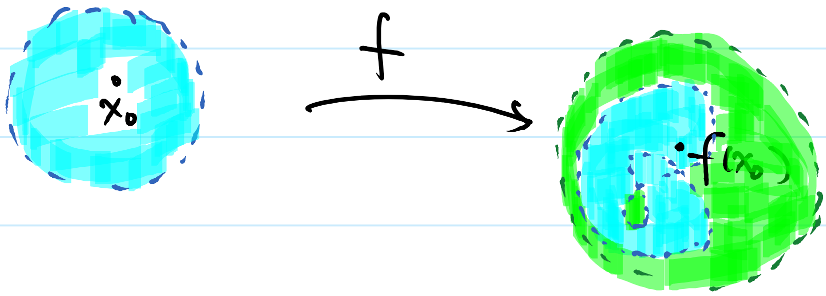
Let us consider the special case when $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ are metric spaces or, in particular $X=\mathbb{R}^n$, $Y=\mathbb{R}^m$.

Let $x_0 \in X$, $f(x_0) \in Y$. Recall that f is continuous at x_0 if $\forall\ \varepsilon>0\ \exists\ \delta>0$ such that if

$$\underbrace{\|x-x_0\|<\delta}\ \text{then}\ \underbrace{\|f(x)-f(x_0)\|<\varepsilon}$$

$$\underbrace{\text{if } x\in B_X(x_0,\delta) \text{ then } f(x)\in B_Y(f(x_0),\varepsilon)}$$

$$f(B_X(x_0,\delta)) \subset B_Y(f(x_0),\varepsilon)$$



or $\quad B_X(x_0,\delta) \subset f^{-1}(B_Y(f(x_0),\varepsilon))$

To get rid of metric (or ball), we replace

$B_X(x_0, \delta)$ by $U \in \mathcal{T}_X$ with $x_0 \in U$ and

$B_Y(f(x_0), \varepsilon)$ by $V \in \mathcal{T}_Y$ with $f(x_0) \in V$

A mapping $f: (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous at x_0 if $\forall\ V \in \mathcal{T}_Y$ with $f(x_0) \in V$

$\exists\ U \in \mathcal{T}_X$ with $x_0 \in U$ such that

$f(U) \subset V$ i.e. $U \subset f^{-1}(V)$

Also, it is continuous everywhere if the above is true for all $x_0 \in X$.

Equivalently, $\forall\ V \in \mathcal{T}_Y$, $\underbrace{f^{-1}(V) \in \mathcal{T}_X}$.

$\forall\ x \in f^{-1}(V)$

$\exists\ U \in \mathcal{T}_X$ such that

$x \in U \subset f^{-1}(V)$