Lect02-0114-Nbhd

Monday, January 18, 2016 12:42 AM

Recall that for a nonempty X, a topology $\mathcal{J} \subset \mathcal{P}(X)$ is a set satisfying that it is closed under arbitrary union and finite intersection. Also, a set $G \subset X$ is open if $G \in \mathcal{J}$.

In a way, "open" is an undefined concept. What matters is the "system" that these sets form.

Three examples were given, Discrete Indiscrete, and Cofinite topologies

Qu. What about this? Is it a topology?

Let $X = \mathbb{R}$ and

$$\mathcal{J} = \{\phi, \mathbb{R}\} \cup \{\underbrace{(a-\varepsilon, a+\varepsilon)}_{\text{open intervals}} : a \in \mathbb{R},\ \varepsilon > 0\}$$

Answer is NO. because $(1,2) \cup (5,6) \neq (a-\varepsilon, a+\varepsilon)$

Though it is closed under finite intersection.

Example $X = \mathbb{R}$ and $\mathcal{J} = \{\phi, \mathbb{R}, [1,3], [2,4], [2,3], [1,4]\}$

Now, $\mathcal{J}$ is a topology for $\mathbb{R}$.

$[1,3]$ is open, though it is closed in the normal topology

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Metric Topology Let (X,d) be a metric space.

For $A \subset X$, define

$$\mathring{A} = \{ a \in A : \exists\, \delta > 0 \ B(a,\delta) \subset A \}$$

$$\mathcal{J} = \{ G \subset A : \mathring{G} = G \}$$

$$\underset{\text{exercise}}{=} \{ \text{all possible unions of balls} \}$$

Recall that on $\mathbb{R}^n$, we have metrics

$$\|x-y\|_p = \left[\sum_{k=1}^{n} |x_k - y_k|^p \right]^{1/p}, \quad p \geq 1 \text{ and}$$

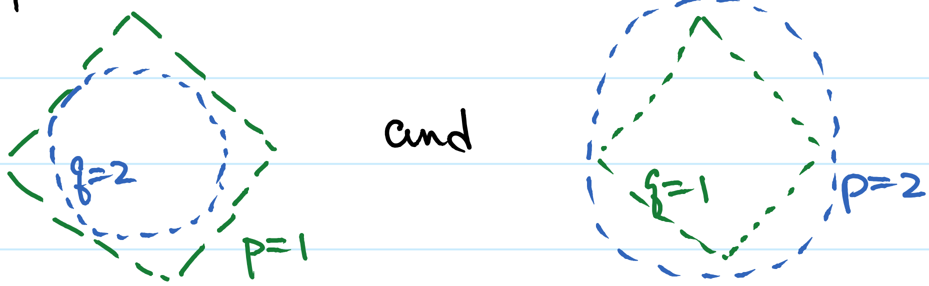
$$\|x-y\|_\infty = \max\{ |x_k - y_k| : k = 1, \cdots, n \}$$

The balls defined by these metrics are convex.

Fact. Their metric topologies are the same.

Crucial step to prove this fact: given any ε-Ball of p-metric, it contains a δ-Ball of q-metric for some $\delta > 0$.

The pictures are



Now you see that this argument still works if the balls are non-convex !!

We mentioned that the Discrete Topology $\mathcal{T} = \mathcal{P}(X)$ actually comes from the discrete metric. Also, the Indiscrete Topology $\{\emptyset, X\}$ does not come from any!!

Qu. How to conclude such a result?

Obviously, one cannot try all possible metric. Standard logical argument:

Metric topology is "beautiful" and Indiscrete Topology is "ugly", so it does not come from any metric.

Qu. What sort of "beauty" we are talking?

Given a metric d and any $x \neq y \in X$, we have $d(x,y) = r > 0$.

Then the balls $B(x, \frac{r}{3})$, $B(y, \frac{r}{3})$ are disjoint (Exercise. Needs Δ-inequality). Writing $U \in B(x, \frac{r}{3})$, $V = B(y, \frac{r}{3})$, we have the Hausdorff property (or called T_2):

For $x \neq y \in X$, $\exists$ $U, V \in \mathcal{T}$ such that

$x \in U$, $y \in V$, and $U \cap V = \emptyset$

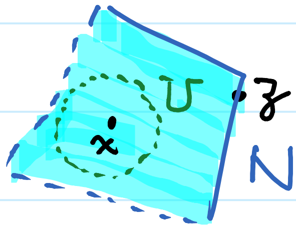
Fact. Every metric topology is Hausdorff

Exercise (a) Indiscrete topology is not Hausdorff.

(b) Is cofinite topology Hausdorff?

Given a topology $\mathcal{J}$ for X, written $(X, \mathcal{J})$. $x \in X$ and $N \subset X$. We say that N is a neighborhood of x if $\exists\, U \in \mathcal{J}$, $x \in U \subset N$.

In this situation, we also say that x is an interior point of N, denoted $x \in \mathring{N}$ or $\mathrm{Int}(N)$



In the picture, $x \in \mathring{N}$, $z \in N$ but $z \notin \mathring{N}$

For each $x \in X$, we may define $\mathcal{N}_x \subset \mathcal{P}(X)$ by $\mathcal{N}_x = \{N \subset X : N \text{ is a nbhd of } x\}$

There may be many sets containing x and form $\mathcal{N}_x$. The collection $\mathcal{N}_x$ satisfies

(N1) $\forall\, N \in \mathcal{N}_x$, $x \in N$ (Obvious)

(N2) $\forall\, M, N \in \mathcal{N}_x$, $M \cap N \in \mathcal{N}_x$ (Semi-obvious)

(N3) If $N \in \mathcal{N}_x$ and $N \subset M$ then $M \in \mathcal{N}_x$ (obvious)

(N4) For $N \in \mathcal{N}_x$, temporarily let $\overset{\odot}{N} = \{y \in N : N \in \mathcal{N}_y\}$ then $\overset{\odot}{N} \in \mathcal{N}_x$ (Exercise)

In the history of "topology", mathematicians tried different ways to develop the concept. Nowadays, we settle on $\mathcal{J} \subset \mathcal{P}(X)$ which is closed under arbitrary union and finite intersection. Neighborhood System is another approach.

A Neighborhood System of X is a mapping

$$x \longmapsto \mathcal{U}_x : X \longrightarrow \mathcal{P}(X)$$

such that the sets $\mathcal{U}_x$ satisfy (N1) to (N4).

Theorem. Each neighborhood System defines a topology $\mathcal{J}$ such that for each $x \in X$, if we define $\mathcal{N}_x$ as before, then

$$\mathcal{N}_x = \mathcal{U}_x$$

we will clarify more later

Qu. Let $X = \mathbb{R}$

$$\mathcal{U}_x = \left\{ \left(x - \frac{1}{n}, x + \frac{1}{n}\right) : 1 \leq n \in \mathbb{Z} \right\}$$

Does $\mathcal{U}_x$ satisfy (N1) to (N4) ?

Do you think that a topology is determined ?