Lect01-0112-Topology

Wednesday, January 13, 2016 7:19 PM

Recall the knowledge of metric space.

How do we define an open set G a metric space X?

Two methods to define

* G is a union of balls, or
* Every point of G is an interior point of G. Mathematically written $\mathring{G} \supseteq G$

⇨ Go to Note01-metric for more review

Since interior point is defined using balls also, the concept of balls is essential in metric spaces.

How does a typical ball $B(0,r)$ look like, where $0=(0,0)\in\mathbb{R}^2$ and the metric

$$d(x,y) = \|x-y\|_p = \left[\sum_{k=1}^{2} |x_k - y_k|^p\right]^{1/p},\ p \geq 1$$

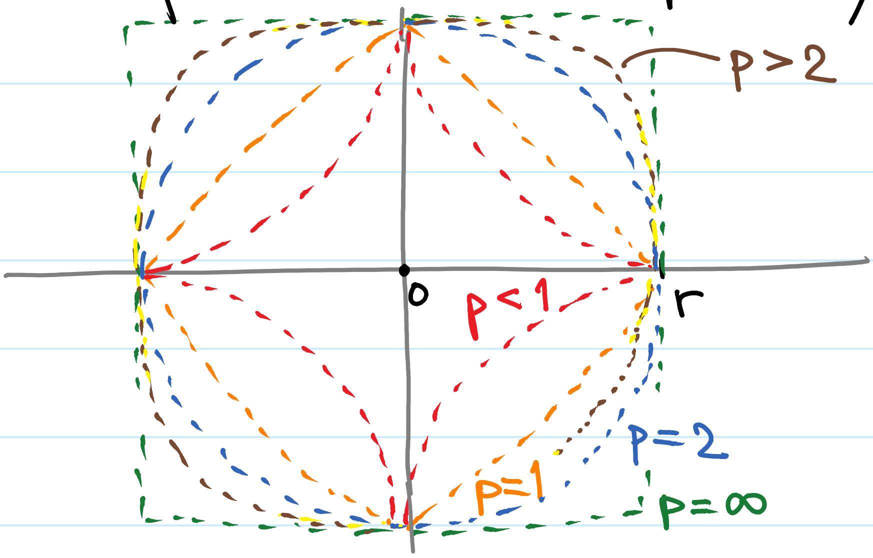
for different values of p?

Note that when $p=2$, it is our usual distance concept in $\mathbb{R}^2$.

Also, we may consider

$$\|x-y\|_\infty = \max\{|x_1-y_1|, |x_2-y_2|\}$$

Pictures of $B(0,r) \subset \mathbb{R}^2$ for $\|x-y\|_p$



One will see that $B(0,r)$ is convex for $p \geq 1$, which is related to Δ-inequality. In the case of $p<1$, Δ-inequality does not hold and $B(0,r)$ is non-convex.

When $r \to 0$, these balls $B(0,r)$ shrink to a point. This is related to the process of taking limit $x \to 0$.

Clearly, it works even $B(0,r)$ is non-convex. In other words, we do not need Δ-inequality nor metric to make sense in $x \to 0$.

The crucial thing is a system of open sets

Definition. Let X be nonempty. A set $\mathcal{T} \subset \mathcal{P}(X)$ is a topology for X if it satisfies

(T1) Any union of sets in $\mathcal{T}$ is still in $\mathcal{T}$.

(T2) Any finite intersection of sets in $\mathcal{T}$ is still in $\mathcal{T}$.

(T3) $\phi \in \mathcal{T}$ and $X \in \mathcal{T}$.

For (T1) and (T2), we often simply say:

A topology $\mathcal{T}$ is closed under arbitrary union and finite intersection.

Logically, (T1) and (T2) $\Longrightarrow$ (T3)

Math Notation.

(T1) For all $\{G_\alpha : \alpha \in I\} \subset \mathcal{T}$, $\bigcup_{\alpha \in I} G_\alpha \in \mathcal{T}$

or For all $\mathcal{A} \subset \mathcal{T}$, $\bigcup \mathcal{A} \in \mathcal{T}$

(T2) For all $G_1, \cdots, G_n \in \mathcal{T}$, $G_1 \cap \cdots \cap G_n \in \mathcal{T}$

or For all finite $\mathcal{F} \subset \mathcal{T}$, $\bigcap \mathcal{F} \in \mathcal{T}$

In this notation, one may prove

$\bigcup \phi = \phi$ and $\bigcap \phi = X$, $\therefore$ (T3) follows.

Thursday, January 14, 2016 11:45 AM

We start with two extreme examples. They usually serve for checking certain concept

Discrete Topology. Let $\mathcal{J} = \mathcal{P}(X)$.

Clearly (T1), (T2), (T3) are satisfied.

Indiscrete Topology Let $\mathcal{J} = \{\phi, X\}$

(T3) is obvious

(T1) and (T2) are logically satisfied.

Co-finite Topology Let

$$\mathcal{J} = \{ G \subset X : X \setminus G \text{ is finite} \} \cup \{\phi\}$$

It is clear that (T3) is satisfied

Exercise. Logically, work out that (T1) and (T2) are valid.

In the proof, you will use De Morgan's

$$X \setminus \bigcup_{\alpha \in I} G_\alpha = \bigcap_{\alpha \in I} (X \setminus G_\alpha)$$

$$X \setminus \bigcap_{k=1}^{n} G_k = \bigcup_{k=1}^{n} (X \setminus G_k)$$

And see that only finite intersection works

Qu. What happens if X itself is finite?