# Probability Distribution (Continuous random variable)

From Discrete to Continuous case :

e.g. A certain traffic light remains red for 50 seconds at a time.
Andy arrives (at random) at the light and finds it red.
Let $X$ be the random variable denotes the waiting time of Andy.

Q : P( Andy has to wait for at least 10 sec ) ?

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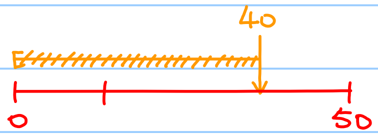



Arrive at time $x$

Waiting time $= 50 - x$

A : Required probability $= \dfrac{40}{50} = \dfrac{4}{5}$

$$\frac{\text{length of the interval } [0,40]}{\text{length of the interval } [0,50]}$$

Q : P( Andy has to wait for exactly 10 sec ) ?

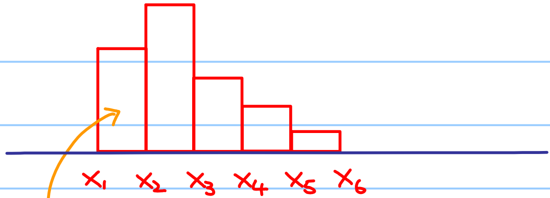
A : Required probability $= \frac{0}{50} = 0$ ( strange ! )

$$\frac{\text{length of the interval } \{40\}}{\text{length of the interval } [0,50]}$$ ← single point.

Conclusion : We cannot use the same treatment for continuous case !

Basically, we cannot talk about the probability for a particular value, but we can talk about the probability for an interval.

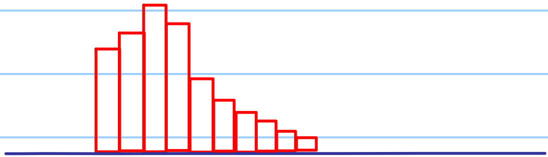
Think :



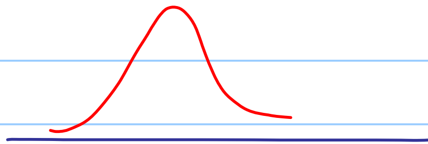

Area $= P(x_1 \le X \le x_2)$

$\therefore$ Summing up the areas of the rectangles $= 1$

$\Downarrow$ More partitions



taking limit $\Longrightarrow$



Probability Distribution (Continuous random variable)

$X$ : Continuous random variable

A probability distribution (or probability density function pdf) is a function $f$ such that

1) $f(x) \geq 0$

2) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

and under this distribution, we have

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

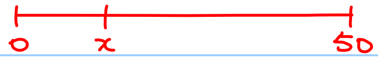
e.g. A certain traffic light remains red for 50 seconds at a time.
Andy arrives (at random) at the light and finds it red.

Sample space $= S = [0, 50]$

Let $X$ be the random variable that denotes the waiting time of Andy

$X : S \to \mathbb{R}$

$$X(x) = \begin{cases} 50 - x & \text{if } 0 \leq x \leq 50 \\ 0 & \text{otherwise} \end{cases}$$

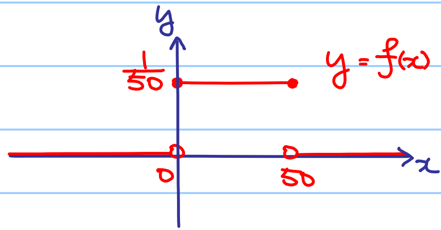



Arrive at time $x$

Waiting time $= 50 - x$

image of $X = [0, 50]$
which is an uncountably infinite
$\therefore X$ is a continuous random variable.

$$f(x) = \begin{cases} \frac{1}{50} & \text{if } 0 \le x \le 50 \\ 0 & \text{otherwise.} \end{cases}$$




$$P(10 \le X \le 50) = \int_{10}^{50} f(x)\,dx$$

$$= \left[\frac{1}{50}x\right]_{10}^{50} = \frac{40}{50} = \frac{4}{5}$$

Remark:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{othewise} \end{cases}$$

This probability density function is called a uniform distribution.

(More examples of probability density function)

## Exponential Density Function

Let $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ , where $\lambda > 0$

Verify $f$ is a probability distribution!

① $f(x) \geq 0$

Easy to check!

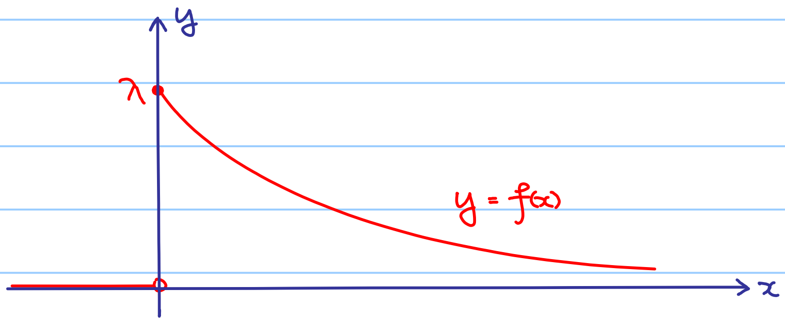
② $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{+\infty} \lambda e^{-\lambda x}\,dx$$

$$= \lim_{N\to+\infty} \int_0^N \lambda e^{-\lambda x}\,dx$$

$$= \lim_{N\to+\infty} \left[-e^{-\lambda x}\right]_0^N$$

$$= \lim_{N\to+\infty} 1-e^{-\lambda N} = 1$$



e.g. (P. 846, Example 11.2.4)

Let $X$ be a random variable that measures the duration of cell phone calls in a certain city, and assume that $X$ has an exponential distribution with density

$$f(t) = \begin{cases} 0.5e^{-0.5t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

where $t$ denotes the duration (in minutes) of a randomly selected call.

a) Find the probability that a randomly selected call lasts no more than 1 minute.

b) Find the probability that a randomly selected call will last at least 2 minutes.

a) $P(0 \leq X \leq 1) = \int_0^1 f(t)\,dt = \int_0^1 0.5e^{-0.5t}\,dt \overset{\text{Ex.}}{=} \cdots = 1 - e^{-0.5} \approx 0.3935$

b) $P(X \geq 2) = \int_2^{+\infty} f(t)\,dt = \lim_{N \to +\infty} \int_2^N 0.5e^{-0.5t}\,dt \overset{\text{Ex.}}{=} \cdots = e^{-1} \approx 0.3679$

Remark:

1) $P(X \geq 2) = P(X > 2)$ why ?

2) How to obtain the probability distribution ?

Even we conduct a survey to collect all data,

we have finitely many calls (but a big number)

then the sample space $= S =$ set of all calls

$X : S \to \mathbb{R}$ is a random variable that denotes the duration

image of $S$ is still finite.

But it can be approximated by the $X$ in the question.

Cumulative distribution function (cdf)

$F(x) = P(X \leq x)$

$= \text{Probability of getting value} \leq x$

Q : What is the relation between the pdf and cdf ?

A : $F(x) = P(X \leq x)$

$= \int_{-\infty}^{x} f(t)\,dt$

e.g. $$f(t) = \begin{cases} 0.5e^{-0.5t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt \qquad \text{if } x \geq 0$$

$$= \int_{0}^{x} 0.5e^{-0.5t}\,dt$$

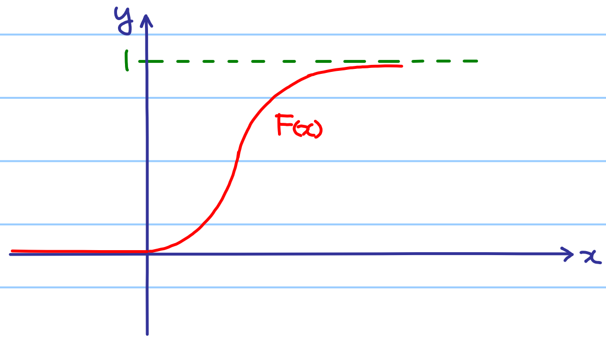
Ex.

$$= \cdots = 1 - e^{-0.5x}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt \qquad \text{if } x < 0$$

$$= \int_{-\infty}^{x} 0\,dt = 0$$

$$\therefore \quad F(x) = \begin{cases} 1 - e^{-0.5x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$



Given $f(x)$, we can find $F(x)$ by

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

But, if $F(x)$ is given, how to find $f(x)$?

Ans : Simply by FTC,

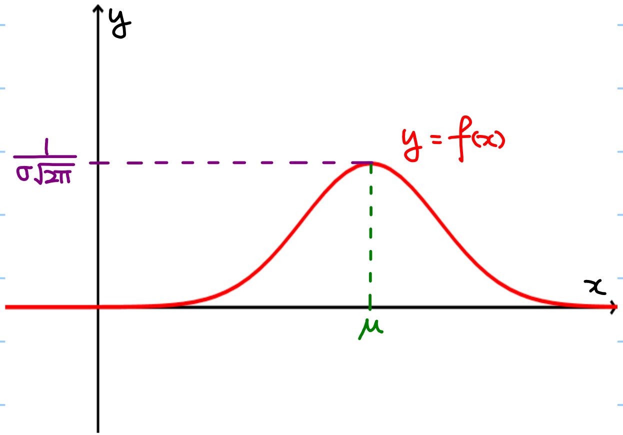
$$F'(x) = f(x) \qquad \text{(if exists)}$$

i.e. cdf is an antiderivative of pdf.

## Normal distribution

A normal random variable with the density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$




bell - shape

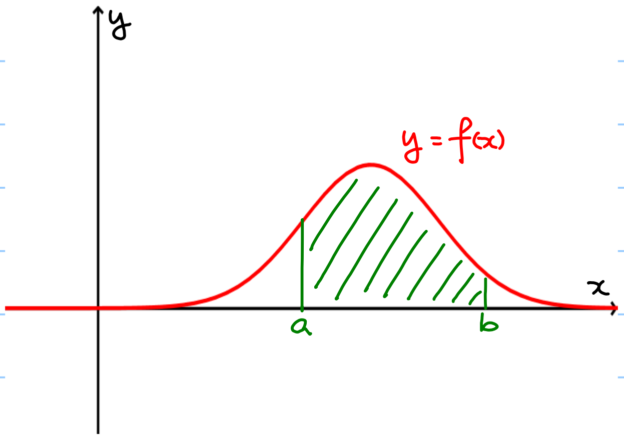
distribution of height, weight, and etc can be approximated by normal distribution.

FACT : ① $f(x) \geq 0$

② $\int_{-\infty}^{+\infty} f(x)\,dx = 1$ ← hard to show.

③ $\mu$ and $\sigma$ are exactly the mean and standard derivation respectively (discuss later)

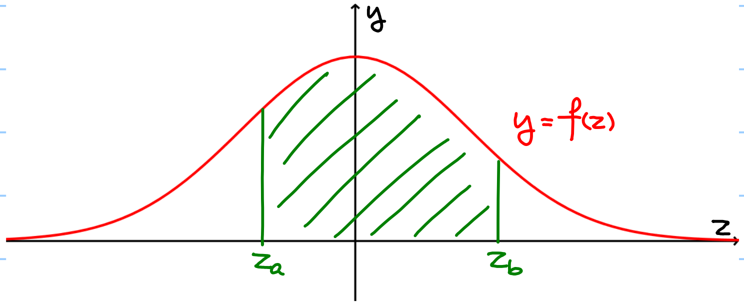
How to find $P(a \leq X \leq b) = \int_a^b f(x)\,dx$ ?



Standard score

let $z = \frac{x-\mu}{\sigma}$

After transformation $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$




$z_a = \frac{a-\mu}{\sigma}$ $z_b = \frac{b-\mu}{\sigma}$

$$\int_a^b f(x)\,dx = \int_{z_a}^{z_b} f(z)\,dz \qquad \text{(Change of variable)}$$

We have a table to obtain an approximation of it.

Idea: Normal distributions with different $\mu, \sigma$

$\Downarrow$ transformation $z = \frac{x-\mu}{\sigma}$

Make it to be the standard one $(\mu = 0, \sigma = 1)$

And then check the table

For detail, refer to section 11.4, very useful!