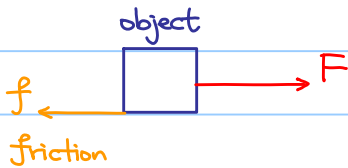


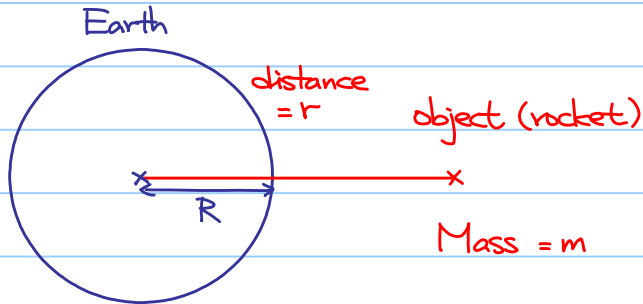
Work done (energy)



moves with constant velocity $\Rightarrow F = f$

distance traveled = s

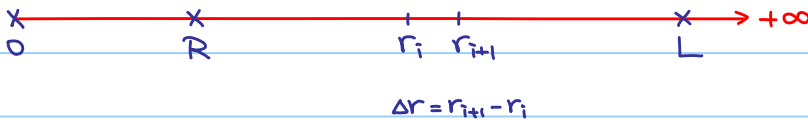
Work done by F against friction = $F \cdot s$
(energy)



Gravitational Force = $\frac{GMm}{r^2}$

Remark :

$G = 6.67 \times 10^{-11} \text{ (m}^3\text{kg}^{-1}\text{s}^{-2}\text{)}$



Energy to bring the rocket from r_i to r_{i+1}

$\approx \frac{GMm}{r_i^2} \Delta r$

Energy to bring the rocket from R to L

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{GMm}{r_i^2} \Delta r$$

$$= \int_R^L \frac{GMm}{r^2} dr$$

$$= \left[\frac{-GMm}{r} \right]_R^L$$

$$= GMm \left(-\frac{1}{L} + \frac{1}{R} \right)$$

Energy to bring the rocket from R to $+\infty$ (Escape from the Earth)

$$= \int_R^{+\infty} \frac{GMm}{r^2} dr$$

$$= \lim_{L \rightarrow +\infty} \int_R^L \frac{GMm}{r^2} dr$$

$$= \lim_{L \rightarrow +\infty} \left[\frac{-GMm}{r} \right]_R^L$$

$$= \lim_{L \rightarrow +\infty} GMm \left(-\frac{1}{L} + \frac{1}{R} \right)$$

$$= \frac{GMm}{R}$$

Recall: Conservation of energy

⇒ initial kinetic energy = energy for the rocket to escape

$$\frac{1}{2}mv^2 = \frac{GMm}{R}$$

$$v = \sqrt{\frac{2GM}{R}} \text{ called escape velocity}$$

i.e. minimum velocity to launch the rocket

Differential Equation

A differential equation is an equation that involves some function of one or more variables with its derivatives.

e.g. $\frac{dy}{dx} = 3x^2 + 5$, $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 7x + 3$

$$3 \frac{\partial f(x,y)}{\partial x} + 2 \frac{\partial f(x,y)}{\partial y} = 3xy , \dots$$

What we do here :

First order ordinary differential equation (1st order ODE)

Ordinary : Single variable , i.e. y depends on x only.

1st order : Involving derivatives up to 1st order , i.e. $\frac{dy}{dx}$ at most

i.e. $F(x, y, \frac{dy}{dx}) = 0$.

Solving the ODE $F(x, y, \frac{dy}{dx}) = 0$ means finding $y(x)$.

$$\cdot \frac{dy}{dx} = g(x)$$

$$\text{simplest one: } y(x) = \int g(x) dx$$

$$\text{e.g. solve } \frac{dy}{dx} = 3x^2 + 5$$

$$y = \int 3x^2 + 5 dx$$

$$y = x^3 + 5x + C$$

• separable equation $\frac{dy}{dx} = \frac{h(x)}{g(y)}$

method : $g(y)dy = h(x)dx$

$$\int g(y)dy = \int h(x)dx$$

e.g. $\frac{dy}{dx} = \frac{2x}{y}$

$$y^2 dy = 2x dx$$

$$\int y^2 dy = \int 2x dx$$

$$\frac{1}{3}y^3 + C_1 = x^2 + C_2 \quad \text{let } C' = C_2 - C_1$$

(This step can be skipped.)

$$\frac{1}{3}y^3 = x^2 + C'$$

$$y^3 = 3x^2 + C \quad (\text{let } C = 3C')$$

$$y = (3x^2 + C)^{\frac{1}{3}}$$

e.g. Spread of rumor.

Population of a school = 200

t : time (day)

Assumption: each one who knows the rumor would talk to 5 people each day

$x(t)$: number of people who know the rumor at time t .

$$x(0) = 20$$

$\frac{dx}{dt}$ = rate of increase of people who know the rumor.

= 5 × # people who know the rumor at time t

× probability of meeting a person who does NOT know the rumor

$$= 5x \left(\frac{200-x}{200} \right)$$

$$= \frac{x(200-x)}{40}$$

$$\frac{dx}{dt} = \frac{x(200-x)}{40}$$

$$\int \frac{40}{x(200-x)} dx = \int dt$$

$$\frac{1}{5} \ln \left| \frac{x}{200-x} \right| = t + C$$

put $t=0, x=20 : C = \frac{1}{5} \ln \frac{1}{q}$

$$\frac{1}{5} \ln \left| \frac{x}{200-x} \right| = t + \frac{1}{5} \ln \frac{1}{q}$$

$$\frac{200-x}{x} - 1 = q e^{-5t}$$

$$x = \frac{200}{1 + q e^{5t}}$$

Remark:

1) $\lim_{t \rightarrow +\infty} x(t) = \lim_{t \rightarrow +\infty} \frac{200}{1 + q e^{5t}} = 200$. Eventually, everybody knows.

2) How long does it take so that half of people know the rumor?

e.g. Parachute

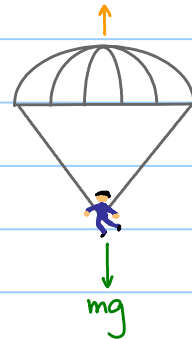
Recall: Newton 2nd Law of motion

$$F = ma = m \frac{dv}{dt}$$

Net force mass acceleration

$$m \frac{dv}{dt} = mg - kv$$

air resistance = $-kv$



gravitational force

$$g = 9.8 \text{ ms}^{-2}$$

When $t=0$, $v=v_0$

v_0 is called initial velocity.

$$m \frac{dv}{dt} = mg - kv$$

$$\frac{m}{mg - kv} dv = dt$$

$$\int \frac{m}{mg - kv} dv = \int dt$$

$$-\frac{m}{k} \ln |mg - kv| = t + C_1$$

$$|mg - kv| = C_2 e^{-\frac{k}{m}t} \quad (C_2 = e^{-\frac{kC_1}{m}} > 0)$$

$$mg - kv = \pm C_2 e^{-\frac{k}{m}t}$$

$$v = \frac{mg}{k} \mp \frac{C_2}{k} e^{-\frac{k}{m}t}$$

$$v(0) = v_0 \Rightarrow v_0 - \frac{mg}{k} = \mp \frac{C_2}{k} \quad (\text{Which sign should we pick?})$$

$$v = \frac{mg}{k} + \frac{C_2}{k} e^{-\frac{k}{m}t}$$

Case 1 : $v_0 > \frac{mg}{k}$

$$0 < v_0 - \frac{mg}{k} = + \frac{C_2}{k}$$

$\frac{C_2}{k} > 0 \Rightarrow$ we should pick the green one.

$$\therefore C_2 = +(kv_0 - mg) \text{ and } v(t) = \frac{mg}{k} + (v_0 - \frac{mg}{k}) e^{-\frac{k}{m}t}$$

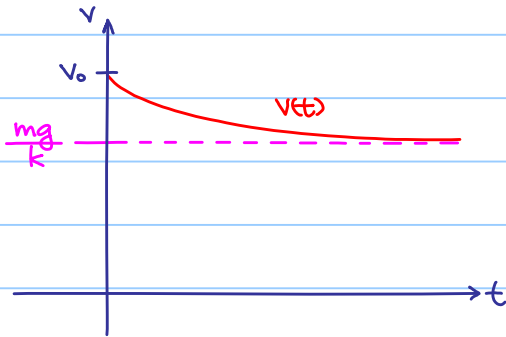
Case 2 : $v_0 < \frac{mg}{k}$

$$0 > v_0 - \frac{mg}{k} = - \frac{C_2}{k}$$

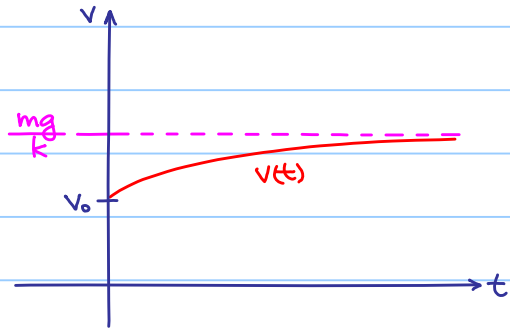
$\frac{C_2}{k} > 0 \Rightarrow$ we should pick the red one.

$$\therefore C_2 = -(kv_0 - mg) \text{ and } v(t) = \frac{mg}{k} - (\frac{mg}{k} - v_0) e^{-\frac{k}{m}t}$$

Case 1 : $v_0 > \frac{mg}{k}$



Case 2 : $v_0 < \frac{mg}{k}$



Remark :

$$1) \lim_{t \rightarrow +\infty} v(t) = \frac{mg}{k}$$

$$\therefore \text{Terminal velocity} = \frac{mg}{k}$$

2) Terminal velocity is independent from the initial velocity v_0 .

First Order Linear Ordinary Differential Equations :

All 1st order linear ODEs are of the form :

$$\frac{dy}{dx} + p(x)y = q(x)$$

Regard $(\frac{d}{dx} + p(x))$ as an operator (differential operator) acting on $y(x)$.

i.e. $(\frac{d}{dx} + p(x))y(x) = \frac{dy}{dx} + p(x)y$

"Linear" means the following properties :

i) $(\frac{d}{dx} + p(x))(y_1(x) + y_2(x)) = (\frac{d}{dx} + p(x))y_1(x) + (\frac{d}{dx} + p(x))y_2(x)$

ii) $(\frac{d}{dx} + p(x))(c \cdot y(x)) = c \cdot (\frac{d}{dx} + p(x))y(x)$, where c is a constant.

(Some advantages in ODE theory if one is linear.)

If $q(x) = 0$, then it is called a homogeneous equation, otherwise it is called inhomogeneous equation.

(Usually, homogeneous equations are easier to be solved.)

Note that if $q(x) = 0$, then

$$\frac{dy}{dx} + p(x)y = 0$$

$$\frac{dy}{dx} = -p(x)y \quad (\text{separable equation})$$

$$\frac{1}{y} dy = -p(x) dx$$

$$\int \frac{1}{y} dy = -\int p(x) dx$$

$$\ln |y| = -\int p(x) dx$$

$$y = \pm \exp(-\int p(x) dx)$$

Note that if $q(x) \neq 0$, then it is no longer separable!

How to solve $\frac{dy}{dx} + p(x)y = q(x)$?

Idea: Can we express $\frac{dy}{dx} + p(x)y$ into $\frac{d}{dx}(\text{?})$?

If yes, then $\frac{dy}{dx} + p(x)y = q(x)$

$$\frac{d}{dx}(\text{?}) = q(x)$$

$$\text{?} = \int q(x) dx$$

Unfortunately, we cannot, but how about multiplying a function $I(x)$ on both sides and try again?

$$I(x) \frac{dy}{dx} + I(x) p(x) y = I(x) q(x)$$

like result obtained by product rule

$$\frac{d}{dx}(I \cdot y) = I \frac{dy}{dx} + \frac{dI}{dx} y$$

∴ The only question left is how to get a function $I(x)$ so that

$$\frac{dI}{dx} y = I(x) p(x) y$$

$$\frac{dI}{dx} = I(x) p(x)$$

$$\int \frac{1}{I} dI = \int p(x) dx$$

$$\ln I = \int p(x) dx$$

Ans: Let $I(x) = e^{\int p(x) dx}$, it is perfect!

$I(x)$ is called an integrating factor.

$$\frac{dy}{dx} + p(x)y = q(x)$$

$$e^{\int p(x)dx} \frac{dy}{dx} + e^{\int p(x)dx} p(x)y = e^{\int p(x)dx} q(x)$$

$$\frac{d}{dx} (e^{\int p(x)dx} y) = e^{\int p(x)dx} q(x)$$

$$\frac{d}{dx} (I(x)y) = I(x)q(x) \quad (\text{Note : } I(x) = e^{\int p(x)dx})$$

$$I(x)y = \int I(x)q(x) dx$$

$$y = \frac{1}{I(x)} \int I(x)q(x) dx$$

e.g. Solve $\frac{dy}{dx} + \frac{y}{x} = e^{-x}$

Note: $p(x) = \frac{1}{x}$ and $q(x) = e^{-x}$

∴

$$I(x) = e^{\int \frac{1}{x} dx}$$

$$= e^{\ln x}$$

$$= x$$

Remark:

$$\int \frac{1}{x} dx = \ln x + C. \text{ Why we choose } C = 0?$$

Check: $C \neq 0$, just multiply both sides by a constant.

Multiply both sides by $I(x) = x$:

$$\frac{dy}{dx} + \frac{y}{x} = e^{-x}$$

$$x \frac{dy}{dx} + y = x e^{-x}$$

$$\frac{d}{dx}(xy) = x e^{-x}$$

$$xy = \int x e^{-x} dx$$

Ex:

↓ Integration by parts

$$= -e^{-x}(x+1) + C$$

↖ DO NOT forget!

$$y = \frac{1}{x} [-e^{-x}(x+1) + C]$$

e.g. $\frac{dy}{dx} = 1 + x + y + xy$

$$\frac{dy}{dx} - (1+x)y = 1+x$$

Note: $p(x) = -(1+x)$

$$I(x) = e^{\int p(x) dx} = e^{-\int (1+x) dx} = e^{-(x + \frac{x^2}{2})}$$

$$\frac{dy}{dx} - (1+x)y = 1+x$$

$$e^{-(x+\frac{x^2}{2})} \frac{dy}{dx} - e^{-(x+\frac{x^2}{2})} (1+x)y = e^{-(x+\frac{x^2}{2})} (1+x)$$

$$\frac{d}{dx} (e^{-(x+\frac{x^2}{2})} y) = e^{-(x+\frac{x^2}{2})} (1+x)$$

$$e^{-(x+\frac{x^2}{2})} y = \int e^{-(x+\frac{x^2}{2})} (1+x) dx$$

$$e^{-(x+\frac{x^2}{2})} y = -e^{-(x+\frac{x^2}{2})} + C$$

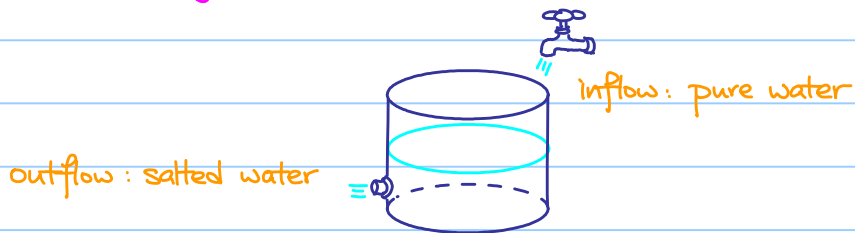
$$y = Ce^{-(x+\frac{x^2}{2})} - 1$$

Ex: By writing $\frac{dy}{dx} = 1+x+y+xy = (1+x)(1+y)$ as a separable equation, try to solve it and compare the results.

Application (setting up ODE)

e.g. (P720, Ex 9.2, Q53)

(Dilution) A tank contains 5 pounds of salt dissolved in 40 gallons of water. Pure water runs into the tank at the rate of 1 gal/min, and the mixture, kept uniform by stirring, runs out at the rate of 3 gal/min.



Set up and solve an initial value problem for the amount of salt $S(t)$ in the tank at time t .

Key point: Everything is clear except this:

What is the rate of change of salt? (Of course, decreasing!)

What happens at time t :

$$\text{Amount of water} = 40 - (3-1)t = 40 - 2t \text{ gal}$$

$$(\because 0 \leq t \leq 20)$$

$$\text{Amount of salt} = S(t) \text{ pound}$$

$$\therefore \text{Concentration} = \frac{S(t)}{40-2t} \text{ pound/gal}$$

Rate of change of salt = - rate of outflow water \times concentration

\uparrow minus sign indicates decreasing

$$\frac{dS}{dt} = - \underbrace{3 \text{ gal/min}}_x \times \frac{S(t)}{40-2t} \text{ pound/gal}_x$$

$$\frac{dS}{dt} = - \frac{3S}{40-2t} \quad (\text{Simply drop } t \text{ in writing } S)$$

(separable equation)

If we rewrite the equation as $\frac{dS}{dt} + \left(\frac{3}{40-2t}\right)S = 0$,

it is in fact a homogeneous 1st order linear ODE

$$\frac{dS}{dt} = -\frac{3S}{40-2t}$$

$$\frac{1}{S} dS = -\frac{3}{40-2t} dt$$

$$\int \frac{1}{S} dS = -3 \int \frac{1}{40-2t} dt$$

$$\ln S = \frac{3}{2} \ln(40-2t) + C'$$

Think: Why $S, 40-2t \geq 0$?

$$S(t) = C(40-2t)^{\frac{3}{2}}$$

Initial condition: $S(0) = 5$

$$\Rightarrow 5 = C \cdot 40^{\frac{3}{2}} \Rightarrow C = 5 \cdot 40^{-\frac{3}{2}}$$

$$\therefore S(t) = \frac{1}{8\sqrt{5}} (20-t)^{\frac{3}{2}} \quad \text{for } 0 \leq t \leq 20$$

Let's look at this:

$$\frac{1}{S} dS = - \frac{3}{40-2t} dt$$

$$\int \frac{1}{S} dS = \int \frac{3}{2t-40} dt$$

$$\ln S = \frac{3}{2} \ln(2t-40) + C'$$

$$S = C(2t-40)^{\frac{3}{2}}$$

When we put $t=0$,

we have $3(-40)^{\frac{3}{2}}$ which is NOT defined!

What's wrong? How to correct?

Let's look at this:

$$\frac{1}{S} dS = - \frac{3}{40-2t} dt$$

$$\int \frac{1}{S} dS = \int \frac{3}{2t-40} dt$$

$$\ln S = \frac{3}{2} \ln(2t-40) + C' \quad \leftarrow \text{Wrong} \quad \because 2t-40 < 0$$

$$S = C(2t-40)^{\frac{3}{2}}$$

we should write:

$$\ln S = \frac{3}{2} \ln|2t-40| + C'$$

When we put $t=0$,

we have $3(-40)^{\frac{3}{2}}$ which is NOT defined!

$$\text{then } \ln S = \frac{3}{2} \ln(40-2t) + C'$$

What's wrong? How to correct?

$$\text{as } |2t-40| = 40-2t$$

e.g. Two kinds of bacteria, X and Y, coexist in environment. Both reproduce at a rate proportional to their numbers, and the constants of proportionality r_x and r_y ($r_y > r_x > 0$) respectively. The environment provides sufficient resource so that no natural deaths occur during the time of investigation. However, each bacterium X kills bacteria Y at the rate of c per unit time. Initially, the population of X and Y are x_0 and ky_0 respectively.

Suppose that after time t , the population of X and Y are $x(t)$ and $y(t)$ respectively. Find $x(t)$ and $y(t)$.

$$\frac{dx}{dt} = r_x \cdot x \quad \Rightarrow \quad \frac{\frac{dx}{dt}}{x} = r_x$$

$\int \frac{1}{x} dx = \int r_x dt$ i.e. relative rate of change of $x(t)$ is a constant.

$$\ln x = r_x t + C_0$$

$$x = e^{r_x t + C_0} = C_0 e^{r_x t}$$

$$x(0) = x_0 \Rightarrow C_0 = x_0$$

$$\therefore x(t) = x_0 e^{r_x t} \quad (\text{exponential growth})$$

$\frac{dy}{dt}$ = rate of change of number of Y

= rate of reproduction of Y - rate of death caused by killing from X

$$= r_y \cdot y - c x$$

$$= r_y \cdot y - c x_0 e^{r_x t}$$

$$\therefore \frac{dy}{dt} - r_y \cdot y = -c x_0 e^{r_x t}$$

(which is a 1st order linear ODE)

$$\text{Integrating factor} = e^{-\int r_y dt} = e^{-r_y t}$$

$$e^{-r_y t} \frac{dy}{dt} - e^{-r_y t} r_y \cdot y = -c x_0 e^{(r_x - r_y)t}$$

$$\frac{d}{dt}(e^{-r_y t} y) = -c x_0 e^{(r_x - r_y)t}$$

$$\begin{aligned} e^{-r_y t} y &= -\int c x_0 e^{(r_x - r_y)t} dt \\ &= -\frac{c x_0}{r_x - r_y} e^{(r_x - r_y)t} + C \end{aligned}$$

$$y(0) = k x_0 \Rightarrow C = k x_0 + \frac{c x_0}{r_x - r_y}$$

$$\therefore e^{-r_y t} y = \frac{c x_0}{r_x - r_y} e^{(r_x - r_y)t} + k x_0 + \frac{c x_0}{r_x - r_y}$$

$$y = \frac{c x_0}{r_x - r_y} e^{r_x t} + \left(k x_0 + \frac{c x_0}{r_x - r_y} \right) e^{r_y t}$$

$$y = \frac{cx_0}{r_x - r_y} e^{r_x t} + \left(k + \frac{c}{r_x - r_y}\right) x_0 e^{r_y t}$$

If Y will extinct at some time t , i.e. $y(t) = 0$ for some $t > 0$.

That means $\frac{cx_0}{r_x - r_y} e^{r_x t} + \left(k + \frac{c}{r_x - r_y}\right) x_0 e^{r_y t} = 0$ has a solution.

$$\underbrace{\frac{cx_0}{r_x - r_y}}_{-ve} \underbrace{e^{r_x t}}_{+ve} = - \left(k + \frac{c}{r_x - r_y}\right) \underbrace{x_0 e^{r_y t}}_{+ve}$$

This equation has solution if and only if

$$-\left(k + \frac{c}{r_x - r_y}\right) < 0$$

$$-k < \frac{c}{r_x - r_y}$$

$$r_x > r_y - \frac{c}{k}$$

Note: $r_x < r_y \Rightarrow r_x - r_y < 0$