e.g. Find $\lim_{n\to\infty} \underbrace{\frac{1^2}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+\cdots+\frac{n^2}{n^3}}_{n \text{ terms}} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3}$

Note: As $n\to\infty$, it is an infinite sum, i.e. summing infinitely many terms.

Algebraic rule does NOT work !!

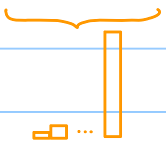
We cannot say: $\lim_{n\to\infty} \frac{1^2}{n^3} = \lim_{n\to\infty} \frac{2^2}{n^3} = \cdots = \lim_{n\to\infty} \frac{n^2}{n^3} = 0$

$\therefore \lim_{n\to\infty} \frac{1^2}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+\cdots+\frac{n^2}{n^3} = 0$

Recall: $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a+(b-a)\frac{i}{n}\right)\cdot\frac{b-a}{n} = \int_a^b f(x)\,dx$

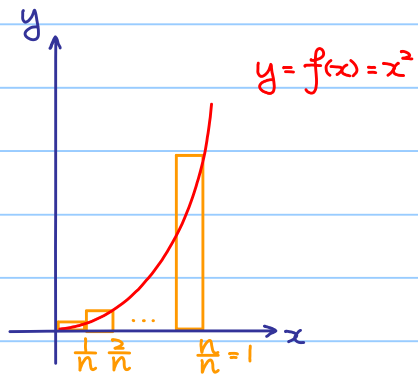
$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^2}\frac{1}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}$



In this case,

$a=0,\ b=1,$



$= \int_0^1 f(x)\,dx$

$= \frac{1}{3}$

Roughly, $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$

$\Downarrow$

$\int_a^b f(x)\,dx$

e.g. Find $\lim_{n\to\infty} \frac{1}{n}\left(e^{1/n}+e^{2/n}+e^{3/n}+\cdots+e^{n/n}\right) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} e^{i/n}$

$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} e^{i/n} = \lim_{n\to\infty} \sum_{i=1}^{n} e^{i/n}\cdot\frac{1}{n}$

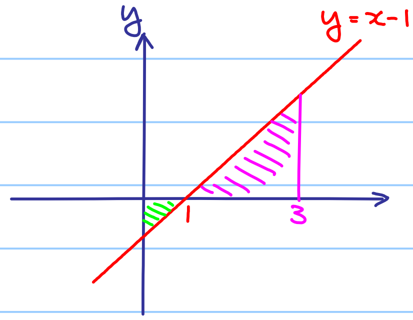
$= \int_0^1 e^x\,dx$

$= [e^x]_0^1$

$= e^1 - e^0$

$= e - 1$

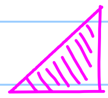
e.g. (NOT area, but signed area)



$$\int_0^1 x-1\,dx = \left[\frac{x^2}{2}-x\right]_0^1 = -\frac{1}{2}$$

$$\int_1^3 x-1\,dx = \left[\frac{x^2}{2}-x\right]_1^3 = 2$$

$$\int_0^3 x-1\,dx = \left[\frac{x^2}{2}-x\right]_0^3 = \frac{3}{2}$$

 −  (Cancellation)

e.g. $\int_{-2}^{3} |x-1|\,dx$

Recall : We can rewrite $|x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ -(x-1) & \text{if } x < -1 \end{cases}$

$$\int_{-2}^{3} |x-1|\,dx = \int_{-2}^{1} |x-1|\,dx + \int_{1}^{3} |x-1|\,dx$$

$$= \int_{-2}^{1} -(x-1)\,dx + \int_{1}^{3} x-1\,dx$$

Ex : $\vdots$

$$= \frac{9}{2} + 2$$

$$= \frac{13}{2}$$

Definite Integral Using Substitution

$$\int_a^b f(u(x))\cdot u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

e.g. $\int_0^1 8x(x^2+1)\,dx$

let $u = x^2+1$

$\frac{du}{dx} = 2x$

$\frac{1}{2}du = x\,dx$

Similar to indefinite integration

$= \int_0^1 8(x^2+1)\ \ x\,dx$

when $x=0$, $u=1$

$x=1$, $u=2$

New ! Don't forget !

$= \int_1^2 8u\ \ \frac{1}{2}du$

Caution !

$= \int_1^2 4u\,du$

$= [2u^2]_1^2$

$= 6$

Remark:

Some may write :

Still 0 and 1

$\int_0^1 8x(x^2+1)\,dx = \int_0^1 4(x^2+1)\,d(x^2+1)$

(as $d(x^2+1) = 2x\,dx$)

$= [2(x^2+1)]_0^1$

$= 6$

(Just the same result !)

e.g. $\int_e^{e^2} \frac{1}{x\ln x}\,dx$

Let $u = \ln x$

$du = \frac{1}{x}dx$

$= \int_1^2 \frac{1}{u}\,du$

when $x=e$, $u=1$

$x=e^2$, $u=2$

$= [\ln u]_1^2$

$= \ln 2 - \cancel{\ln 1}^{\,0}$

$= \ln 2$

Definite Integration Using Integration by Parts

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$$

e.g. $\int_1^e x\ln x\,dx = \int_1^e \ln x \, d\left(\frac{x^2}{2}\right)$

$$= \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2}\,d\ln x$$

$$= \left(\frac{e^2}{2}\ln e - \frac{1}{2}\cancel{\ln 1}^{\,0}\right) - \int_1^e \frac{x}{2}\,dx$$

$$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e$$

$$= \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right)$$

$$= \frac{e^2}{4} + \frac{1}{4}$$

Remark on Computation of Definite Integrals

To compute $\int_a^b f(x)dx$, we need to find an antiderivative $F(x)$ of $f(x)$. then $\int_a^b f(x)dx = F(b) - F(a)$.

Question: But what can we do if we are not able to find an antiderivative $F(x)$?

(OR not interested in finding $F(x)$)

Answer: An approximation may be good enough.

Trapezoidal Rule

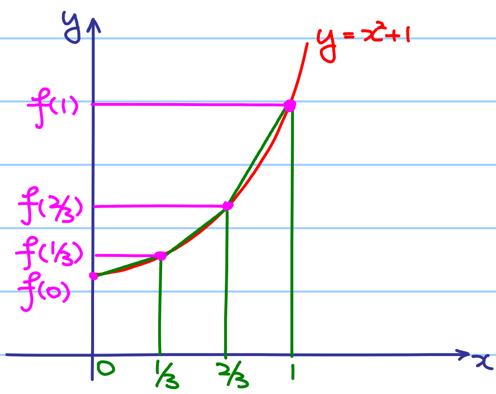
Roughly speaking, approximating by sum of areas of trapeziums.

e.g. $\int_0^1 x^2+1\,dx$ (Approximated by 3 trapeziums)

$$\approx \frac{1}{2}\left[f(0)+f(\tfrac{1}{3})\right]\cdot\frac{1}{3} + \frac{1}{2}\left[f(\tfrac{1}{3})+f(\tfrac{2}{3})\right]\cdot\frac{1}{3} + \frac{1}{2}\left[f(\tfrac{2}{3})+f(1)\right]\cdot\frac{1}{3}$$

$$= \frac{1}{2}\cdot\frac{1}{3}\left[f(0)+2f(\tfrac{1}{3})+2f(\tfrac{2}{3})+f(1)\right]$$

$$= \frac{73}{54} \approx 1.35$$



Compare to $\int_0^1 x^2+1\,dx = \left[\frac{x^3}{3}+x\right]_0^1 = \frac{4}{3} \approx 1.33$

In general, we use n trapeziums (n subintervals)

Usually, more trapeziums we use, better approximation we get!

Refer to section 6.2 of the textbook or deduce the formula yourself:

$$\int_a^b f(x)\,dx = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$ for $i = 0, 1, 2, \cdots, n$.

Application :

e.g. Air pollution

t : time (years)

$L(t)$: Level of carbon monoxide (CO)

Given : $L'(t) = 0.1t + 0.1$ parts per million (ppm) per year.

Q : How much will the pollution change during the next 3 years ?

$$L'(t) = \frac{dL}{dt} = 0.1t + 0.1$$

$$L(t) = \int L'(t)\,dt$$
$$= \int 0.1t + 0.1\,dt$$
$$= 0.05t^2 + 0.1t + C$$

How to determine ? Impossible !

Think carefully : What we need is NOT $L(3)$, but $L(3) - L(0)$!

$$L(t) = 0.05t^2 + 0.1t + C$$

$$L(3) - L(0) = [0.05(3)^2 + 0.1(3) + C] - [0.05(0)^2 + 0.1(0) + C]$$

Cancelled ! No need to determine !

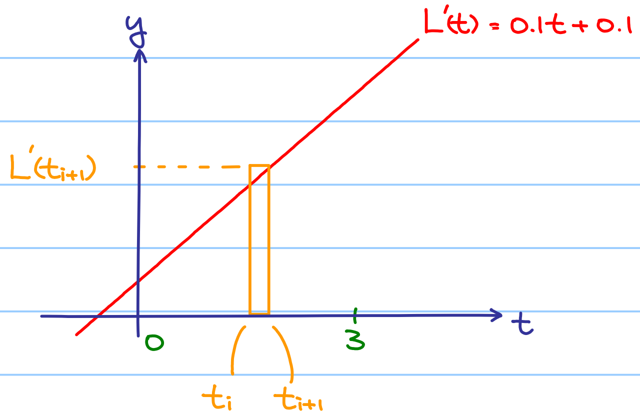
$$= 0.75 \text{ ppm}$$

Compare : $\int_0^3 L'(t)\,dt = \int_0^3 0.1t + 0.1\,dt$

$$= [0.05t^2 + 0.1t]_0^3$$
$$= [0.05(3)^2 + 0.1(3)] - [0.05(0)^2 + 0.1(0)]$$
$$= 0.75 \text{ ppm}$$

(same !)

Geometrically :



Area of the rectangle $= L'(t_{i+1}) \times (t_{i+1} - t_i) \approx$ change of CO between t_i and t_{i+1}

speed of CO change at t_{i+1} duration

$\int_0^3 L'(t)\,dt$ = Area of the region under the graph $L'(t)$ over the interval $0 \le t \le 3$.

(= summing up the areas of all rectangles and taking limit $\lim_{n \to \infty} \sum_{i=1}^{n} f(t_i) \Delta t$)

= change of CO from $t = 0$ to $t = 3$.

Summary :

Nothing, but the result followed from Fundamental Theorem of Calculus :

$$\underbrace{F(b) - F(a)}_{\text{change of } F \text{ from } x=a \text{ to } x=b} = \int_a^b \underbrace{F'(x)}_{\text{rate of change of } F \text{ with respect to } x.}\,dx$$

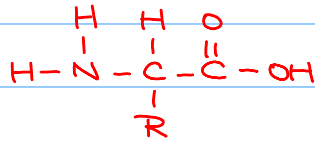
e.g. Change in Biomass

t : time (hour)

$m(t)$: mass of a protein

protein $\longrightarrow$ amino acid

cause a decrease in mass at a rate :



$$\frac{dm}{dt} = \frac{-2}{t+1} \quad g/hr$$

decrease in mass of a protein from $t=2$ to $t=5$

$= m(5) - m(2)$

$= \int_2^5 \frac{dm}{dt}\, dt$

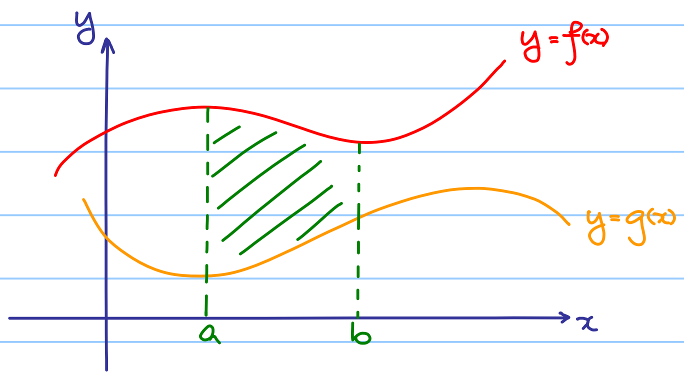
$= \int_2^5 \frac{-2}{t+1}\, dt$

$= \left[-2\ln|t+1|\right]_2^5$

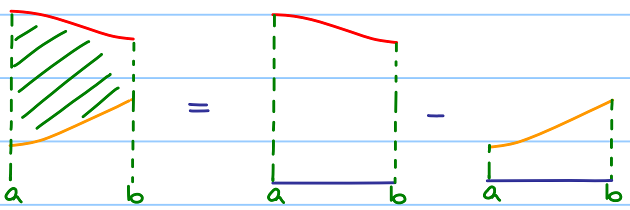
$= -2\ln 6 + 2\ln 3$

$= -2\ln 2$ (negative sign indicates a decrease)

Area Between Curves :



Area of shaded region $= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$



e.g. Find the area bounded by $y=x^2$ and $y=x^3$.

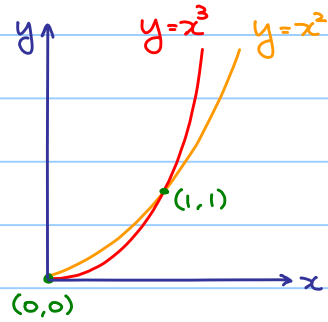
Step 1 : Solve $\begin{cases} y=x^2 \\ y=x^3 \end{cases}$

$$x^3=x^2$$
$$x^2(x-1)=0$$
$$x=0 \quad \text{or} \quad 1$$

(Remark : No need to solve y)

Step 2 : Note when $0 \le x \le 1$, $x^3 \le x^2$

$$\text{Area} = \int_0^1 x^2 - x^3\,dx$$
$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$
$$= \frac{1}{12}$$



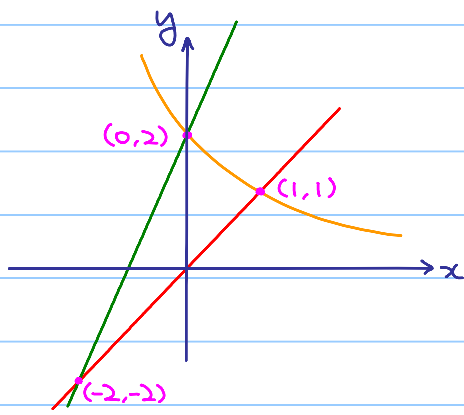
e.g. Find the area bounded by

$y=f(x)=x$, $y=g(x)=\frac{2}{x+1}$ and $y=h(x)=2x+2$

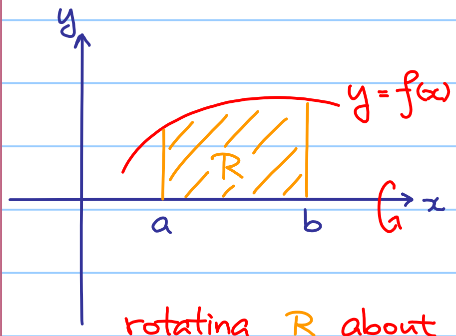
$$\text{Area} = \int_{-2}^{0} h(x)-f(x)\,dx + \int_0^1 g(x)-f(x)\,dx$$

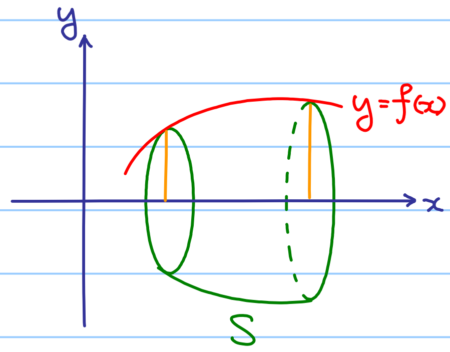
Ex : ⋮

Ans :
$$= 2 + \left(-\frac{1}{2} + \ln 4\right)$$
$$= \frac{3}{2} + \ln 4$$

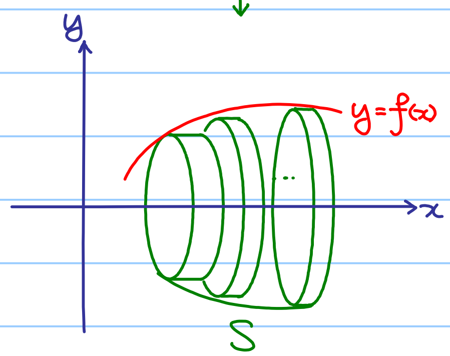


Solids of Revolution and Disk Method.

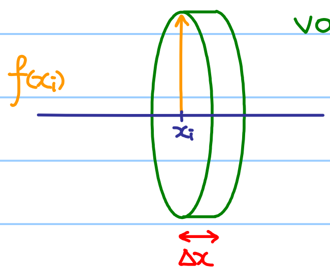




rotating R about x-axis
gives a solid S



volume of S
is approximated
by solid disks

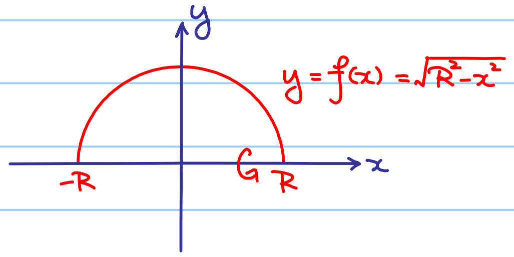


volume $\approx \pi[f(x_i)]^2 \Delta x$

Volume of S $= \lim_{n\to\infty} \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x$

$$= \int_a^b \pi[f(x)]^2 dx = \pi \int_a^b [f(x)]^2 dx$$

e.g. $f(x) = \sqrt{R^2 - x^2}$ for $-R \le x \le R$



semi circle

Solid S = a sphere with radius R.

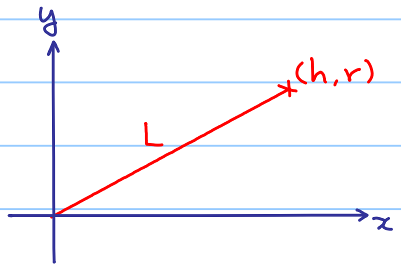
$$\text{Volume of } S = \pi \int_{-R}^{R} [f(x)]^2 \, dx$$

$$= \pi \int_{-R}^{R} R^2 - x^2 \, dx$$

$$= \pi \left[R^2 x - \frac{x^3}{3} \right]_{-R}^{R}$$

$$= \frac{4}{3} \pi R^3$$ (formula in secondary school)

Ex:



a) Find the equation of the straight line L.

b) What is the solid S generated by rotating L about the x-axis?

c) Volume of S = ?

Ans: a) $y = \frac{r}{h} x$

b) a cone with height = h, base radius = r

c) $\frac{1}{3} \pi r^2 h$