Rules for Integrating Common Functions

1) $\int k\,dx = kx + C$, for constant k.

Note: $\frac{d}{dx}(kx+C) = k$

2) $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, for all n except -1.

Note: $\frac{d}{dx}(\frac{1}{n+1}x^{n+1} + C) = x^n$

3) $\int \frac{1}{x}\,dx = \ln|x| + C$ (Interesting when $x<0$)

Note: $\frac{d}{dx}(\ln|x| + C) = \frac{1}{x}$

4) $\int e^x\,dx = e^x + C$

Note: $\frac{d}{dx}(e^x + C) = e^x$

Algebraic Rules For Indefinite Integration

1) $\int kf(x)\,dx = k\int f(x)\,dx$

2) $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Note: 1) $\frac{d}{dx}(\int kf(x)\,dx) = \frac{d}{dx}(k\int f(x)\,dx) = kf(x)$

i.e. $\int kf(x)\,dx$ and $k\int f(x)\,dx$ differ by a constant.

but it is absorbed by $\int$.

2) $\frac{d}{dx}(\int f(x) \pm g(x)\,dx) = \frac{d}{dx}(\int f(x)\,dx \pm \int g(x)\,dx) = f(x) \pm g(x)$.

e.g. $\int 2x^5 - 3x^2 + 7x + 5 \, dx$

$\int dx$ means $\int 1 \, dx$

$= 2\int x^5 dx - 3\int x^2 dx + 7\int x \, dx + 5\int dx$

$\int$ still there,

No need to add $+ C$!

$= 2\left(\frac{x^6}{6}\right) - 3\left(\frac{x^3}{3}\right) + 7\left(\frac{x^2}{2}\right) + 5x + C$

$= \frac{x^6}{3} - x^3 + \frac{7x^2}{2} - 5x + C$

e.g. $\int \frac{x^3 - 5}{x} dx$

$= \int x^2 - \frac{5}{x} dx$

$= \frac{x^3}{3} - 5\ln|x| + C$

e.g. Find a function $F(x)$ such that $F(0) = 3$ and $F'(x) = 2x$.

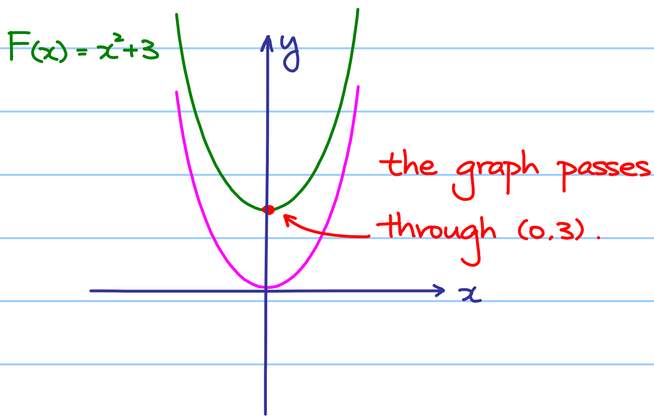
$F'(x) = 2x$

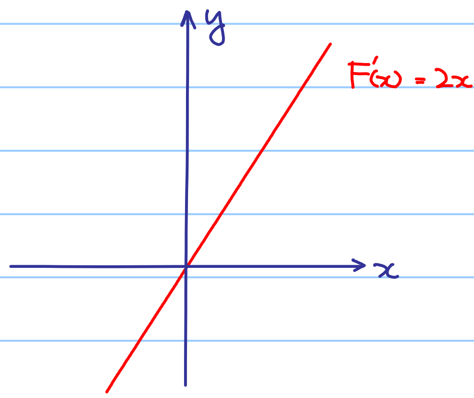
$F(x) = \int 2x \, dx$

$= x^2 + C$

$F(0) = 0^2 + C = 3 \Rightarrow C = 3$

$\therefore F(x) = x^2 + 3$





Integration by Substitution

Question : $\int (2x+1)^{2015} dx = ?$

Hard to integrate by expanding the polynomial.

Solution : Integration by Substitution

Integration by Substitution : $\int f(u(x))\, u'(x)\, dx = \int f(u)\, du$

OR : $\int f(x) \frac{du}{dx} dx = \int f(u)\, du$

proof : $\frac{d}{dx} \int f(u(x))\, u'(x)\, dx = f(u(x))\, u'(x)$

$\frac{d}{dx} \int f(u)\, du = \frac{d}{du} \int f(u)\, du \cdot \frac{du}{dx}$ (Chain Rule)

$= f(u(x)) \cdot \frac{du}{dx}$

$$\frac{d}{dx} \int f(u(x))\, u'(x)\, dx = \frac{d}{dx} \int f(u)\, du$$

$$\therefore \int f(u(x))\, u'(x)\, dx = \int f(u)\, du$$

e.g. $\int (2x+1)^{2015} dx = ?$

Let $u(x) = 2x+1$ $\quad u'(x) = 2$

$f(u) = u^{2015}$ $\quad f(u(x)) = (2x+1)^{2015}$

$$\int (2x+1)^{2015} dx = \frac{1}{2} \int \underbrace{(2x+1)^{2015}}_{f(u(x))} \cdot \underbrace{2}_{u'(x)}\, dx = \frac{1}{2} \int \underbrace{u^{2015}}_{f(u)} du$$

$$= \frac{1}{4032} u^{2016} + C = \frac{1}{4032} (2x+1)^{2016} + C$$

But, usually we write,

$\int (2x+1)^{2015} dx$

$= \int u^{2015} \frac{1}{2} du$

$= \frac{1}{4032} u^{2016} + C$

$= \frac{1}{4032} (2x+1)^{2016} + C$

Let $u = 2x+1$

$\frac{du}{dx} = 2$

$dx = \frac{1}{2} du$

e.g. $\int e^{ax}\,dx$ Let $u = ax$

$= \int e^{u}\cdot\frac{1}{a}\,du$ $\frac{du}{dx} = a$

$= \frac{1}{a}e^{u} + C$ $dx = \frac{1}{a}du$

$= \frac{1}{a}e^{ax} + C$

e.g. $\int 6x(4x^2+3)^7\,dx$ Let $u = 4x^2+3$

$= \int 6(4x^2+3)^7\,x\,dx$ $\frac{du}{dx} = 8x$

$= \int 6u^7\,\frac{1}{8}dx$ $x\,dx = \frac{1}{8}du$

$= \frac{6}{8}\cdot\frac{1}{8}u^8 + C$

$= \frac{3}{32}(4x^2+3)^8 + C$

e.g. $\int \frac{(\ln x)^2}{x}\,dx$, $x>0$

$\int \frac{(\ln x)^2}{x}\,dx$ Let $u = \ln x$

$= \int u^2\,du$ $\frac{du}{dx} = \frac{1}{x}$

$= \frac{1}{3}u^3 + C$ $\frac{1}{x}dx = du$

$= \frac{1}{3}(\ln x)^3 + C$

Question : How to make a guess of $u(x)$?

Integration by Substitution : $\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$

e.g. $\int \frac{(\ln x)^2}{x}\,dx = \int (\ln x)^2\cdot\frac{1}{x}\,dx$ Let $u = \ln x$

Realize the integrand as a product of parts and make a guess of $u(x)$ such that one part can be realized as a function $f(u)$, another part is $u'(x)$

Ex : 1) Show that $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$. Hint : Let $u = ax+b$

2) Evaluate

a) $\int x^3 e^{x^4}\,dx$ Hint : Let $u = x^4$ Ans : $\frac{1}{4}e^{x^4} + C$

b) $\int 6x\sqrt{x^2+3}\,dx$ Hint : Let $u = x^2+3$ Ans : $2(x^2+3)^{\frac{3}{2}} + C$

Integration of Rational Functions :

- $\int \frac{p(x)}{ax+b}\,dx$

By long division, $p(x) = (ax+b)q(x) + R$

$$\frac{p(x)}{ax+b} = q(x) + \frac{R}{ax+b}$$

$$\begin{array}{r} q(x) \\ ax+b \,\overline{)\, p(x)} \\ \vdots \\ \hline R \end{array}$$

Then $\int \frac{p(x)}{ax+b}\,dx = \int q(x) + \frac{R}{ax+b}\,dx$

We know how to integrate!

e.g. $\int \frac{x^2+3x+5}{x+1}\,dx$

$$\begin{array}{r} x+2 \\ x+1 \,\overline{)\, x^2+3x+5} \\ \underline{x^2+x} \\ 2x+5 \\ \underline{2x+2} \\ 3 \end{array}$$

$\therefore\ x^2+3x+5 = (x+1)(x+2)+3$

$$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1}$$

$= \int x+2+\frac{3}{x+1}\,dx$

$= \frac{x^2}{2} + 2x + 3\ln|x+1| + C$

Ex : Evaluate $\int \frac{6x^2-5x+1}{3x-2}\,dx$

Ans : $x^2 - \frac{x}{3} + \frac{1}{9}\ln|3x-2| + C$

- $\int \frac{ax+b}{(r_1x+s_1)(r_2x+s_2)}\,dx$

Express $\frac{ax+b}{(r_1x+s_1)(r_2x+s_2)}$ into the form $\frac{A}{r_1x+s_1} + \frac{B}{r_2x+s_2}$.

Then $\int \frac{ax+b}{(r_1x+s_1)(r_2x+s_2)}\,dx = \int \frac{A}{r_1x+s_1} + \frac{B}{r_2x+s_2}\,dx$

We know how to integrate!

e.g. $\int \frac{5x-7}{x^2-2x-3}\,dx$

Note : $\frac{5x-7}{x^2-2x-3} = \frac{5x-7}{(x-3)(x+1)}$

Suppose $\frac{5x-7}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$

$\Rightarrow\ 5x-7 \equiv A(x+1) + B(x-3)$

$\Rightarrow\ A = 3,\ B = 2.$

$$\int \frac{5x-7}{x^2-2x-3}\,dx = \int \frac{3}{x-3} + \frac{2}{x+1}\,dx = 3\ln|x-3| + 2\ln|x+1| + C$$

Ex : Evaluate $\int \frac{40}{x(200-x)}\,dx$

Ans : $\frac{1}{5}(\ln|x| - \ln|200-x|) + C = \frac{1}{5}\ln\left|\frac{x}{200-x}\right| + C$

- $\int \frac{ax+b}{(px+q)^2}\,dx$

Express $\frac{ax+b}{(px+q)^2}$ into the form $\frac{A}{(px+q)^2} + \frac{B}{px+q}$

Then $\int \frac{ax+b}{(r_1x+s_1)(r_2x+s_2)}\,dx = \int \frac{A}{(px+q)^2} + \frac{B}{px+q}\,dx$

We know how to integrate !

e.g. $\int \frac{2x-1}{(x-2)^2}\,dx$

Suppose $\frac{2x-1}{(x-2)^2} \equiv \frac{A}{(x-2)^2} + \frac{B}{x-2}$

$\Rightarrow 2x-1 \equiv A + B(x-2)$

$\Rightarrow A = 3 \ , B = 2$

$$\int \frac{2x-1}{(x-2)^2}\,dx = \int \frac{3}{(x-2)^2} + \frac{2}{x-2}\,dx = \frac{-3}{x-2} + 2\ln|x-2| + C$$

Ex : Evaluate $\int \frac{4x+2}{(2x-1)^2}\,dx$

Ans : $\frac{-2}{2x-1} + \ln|2x-1| + C$

Remarks :

1) If $\deg p(x) > 1$, $\int \frac{p(x)}{(r_1x+s_1)(r_2x+s_2)}\,dx = ?$

Hint : Long division .

$$\int \frac{p(x)}{(r_1x+s_1)(r_2x+s_2)}\,dx = \int q(x) + \underbrace{\frac{ax+b}{(r_1x+s_1)(r_2x+s_2)}}_{\text{reduced to pervious case !}}\,dx$$

2) If ax^2+bx+c cannot be factorized as a product of two linear factors (i.e. $b^2-4ac<0$) , then $\int \frac{1}{ax^2+bx+c}\,dx = ?$

Unfortunately, we cannot cover this case as it involves trigonometric functions !