e.g A manager of a company, determines that t months after initiating an advertisting compaign, the number of products will be sold is estimated by

$$P(t) = \frac{3}{t+2} - \frac{12}{(t+2)^2} + 5 \quad \text{(thousand)} \quad , \quad t \geq 0.$$

a) Find $P'(t)$ and $P''(t)$.

b) At what time will sales be maximized? What is the maximum level of sales?

c) The manager plans to terminate the advertising compaign when the **sales rate** is minimized. When does it occur?

a) Direct computation:

$$P(t) = \frac{3}{t+2} - \frac{12}{(t+2)^2} + 5$$

$$P'(t) = -\frac{3}{(t+2)^2} + \frac{24}{(t+2)^3} = \frac{18-3t}{(t+2)^3}$$

$$P''(t) = \frac{6}{(t+2)^3} - \frac{72}{(t+2)^4} = \frac{6t-60}{(t+2)^4}$$

(b) Solve $P'(t) > 0$ $\qquad\qquad\qquad\qquad$ $P'(t) < 0$

$\quad \dfrac{18-3t}{(t+2)^3} > 0$ $\qquad\qquad\qquad\qquad$ $\dfrac{18-3t}{(t+2)^3} < 0$

$\quad 18-3t > 0 \quad$ ($\because t \geq 0, \ t+2 > 0$) $\quad 18-3t < 0$

$\qquad\qquad t < 6$ $\qquad\qquad\qquad\qquad\qquad\qquad t > 6$

($P(t)$ is strictly increasing when $t<6$ and strictly decreasing when $t>6$,

$\quad$ $P(t)$ is continuous at $t=6$.)

$\therefore$ $P(t)$ attains maximum when $t=6$. (By 1st derivative check.)

OR: (By observation, $P(t)$ can be differentiated infinitely many times,

$\quad$ so if $P(t)$ attains maximum/minimum at $t=t_0$, we must have $P'(t_0) = 0$,

$\quad$ that's why we consider the equation $P'(t) = 0$.)

$\quad P'(t) = 0$

$\quad \dfrac{18-3t}{(t+2)^3} = 0$

$\qquad\quad t = 6$

(At this moment, we only know $(6, P(6))$ is a stationary point.)

$\quad P''(6) = -\dfrac{24}{8^4} < 0$

$\therefore$ $P(t)$ attains maximum when $t=6$. (By 2nd derivative check.)

Maximum sales level $= P(6) = \dfrac{83}{16}$

(c)   (In fact, we want to minimize $P'(t)$ now!

We apply 1st derivative check to $P'(t)$, i.e. look at $P''(t)$.)

Solve $P''(t) > 0$ $\qquad\qquad$ $P''(t) < 0$

$\dfrac{6t-60}{(t+2)^4} > 0$ $\qquad\qquad$ $\dfrac{6t-60}{(t+2)^4} < 0$

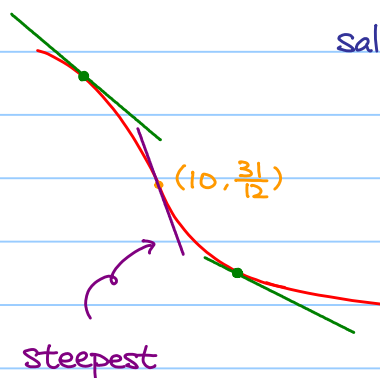$6t - 60 > 0$ $\qquad\qquad$ $6t - 60 < 0$
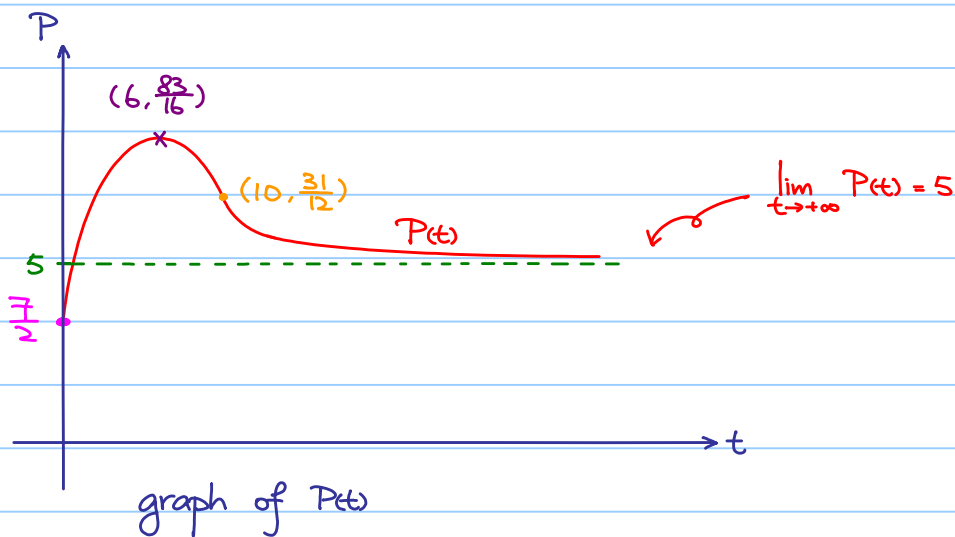
$\qquad t > 10$ $\qquad\qquad\qquad$ $t < 10$

∴ $P'(t)$ attains minimum when $t = 10$. (By 1st derivative check.)

(Note: $(10, P'(10))$ is a point of inflection.)

OR: $P'''(t) = -\dfrac{18}{(t+2)^4} + \dfrac{288}{(t+2)^5} = \dfrac{252-18t}{(t+2)^5}$

$P'''(10) = \dfrac{72}{12^5} > 0$

∴ $P'(t)$ attains minimum when $t = 10$. (By 2nd derivative check.)

P
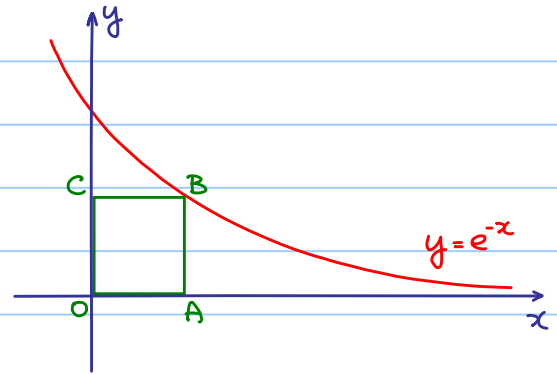
$\left(6, \frac{83}{16}\right)$

$\left(10, \frac{31}{12}\right)$

$\lim\limits_{t \to +\infty} P(t) = 5$

$P(t)$

5

$\frac{7}{2}$

t

graph of $P(t)$

sales rate at t = $P'(t)$

= slope of the tangent line

at $(t, P(t))$

$\left(10, \frac{31}{12}\right)$

steepest

Meaning of minimizing $P'(t)$ in part (c).

e.g. OABC is a rectangle inscribed in the region bounded by the positive coordinate axes and the curve $y = e^{-x}$. Find the maximum area of the rectangle.

Maximize a function!

Dependent variable : Area of OABC, A

Independent variable : x

Area of OABC = OA × AB

$$A = xe^{-x} \qquad x \geq 0$$

$$\frac{dA}{dx} = e^{-x} - xe^{-x}$$

$$= e^{-x}(1-x)$$

$\frac{dA}{dx} > 0$	$\frac{dA}{dx} < 0$
$e^{-x}(1-x) > 0$	$e^{-x}(1-x) < 0$
$1-x > 0$	$1-x < 0$
$1 > x$	$1 < x$

∴ A attains maximum when $x = 1$.

Maximum area of OABC = $A(1) = 1 \cdot e^{-1} = e^{-1}$

Remark: Most Important issue :

1) identifying dependent and independent variable

2) setting up an equation between them

Relative Rates

Suppose x and y are variables related by an equation, but both of them can further be regarded as functions of a third variable t.

(i.e. $x(t)$ and $y(t)$)

(Often : t = time)

Then Implicit differentiation helps to give a relation between $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

e.g. Relation of pollution and population of fish.

  Level of populant = x parts per million (ppm)

  Number of fish = F

  Given $F = \dfrac{32000}{3+\sqrt{x}}$

When there are 4000 fish left in the lake,

the population is increasing at the rate of 1.4 ppm / year.

At what rate is the fish population changing at this time?

  time : t (years)

  $F = 4000$

  $\frac{dx}{dt} = 1.4$ (increasing, $\frac{dx}{dt} > 0$; decreasing, $\frac{dx}{dt} < 0$)

  $\frac{dF}{dt} = ?$ when $\frac{dx}{dt} = 1.4$, $F = 4000$

💡 Idea : Apply implicit differentiation to the equation

  $F = \dfrac{32000}{3+\sqrt{x}}$ and differentiate with respect to t

  $\dfrac{dF}{dt} = \dfrac{d}{dt}\left(\dfrac{32000}{3+\sqrt{x}}\right) = \dfrac{d}{dx}\left(\dfrac{32000}{3+\sqrt{x}}\right)\dfrac{dx}{dt}$ (Apply chain rule)

  $\dfrac{dF}{dt} = \dfrac{-16000}{\sqrt{x}(3+\sqrt{x})^2}\dfrac{dx}{dt}$

  $\frac{dx}{dt} = 1.4$ Oops, $x = ?$

Recall: $F = \dfrac{32000}{3+\sqrt{x}}$, when $x = 4000$
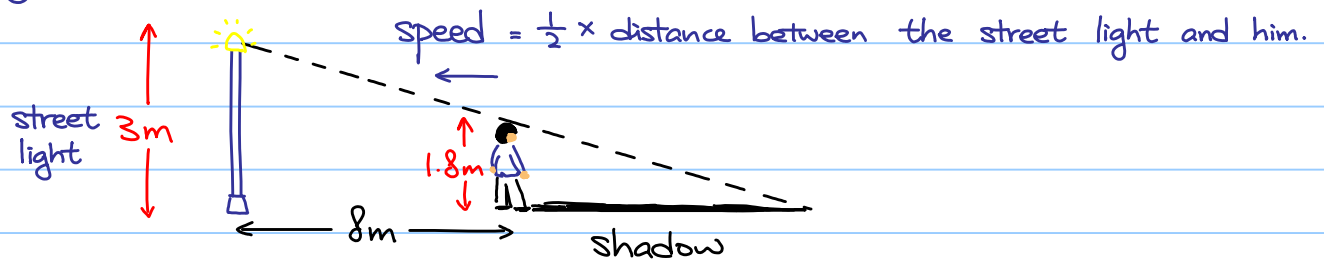
$$4000 = \frac{32000}{3+\sqrt{x}}$$

$$x = 25$$

$$\frac{dF}{dt} = \frac{-16000}{\sqrt{x}\,(3+\sqrt{x})^2} \frac{dx}{dt} = \frac{-16000}{\sqrt{25}\,(3+\sqrt{25})^2} \times 1.4 = -70 \quad \text{(fish per year)}$$
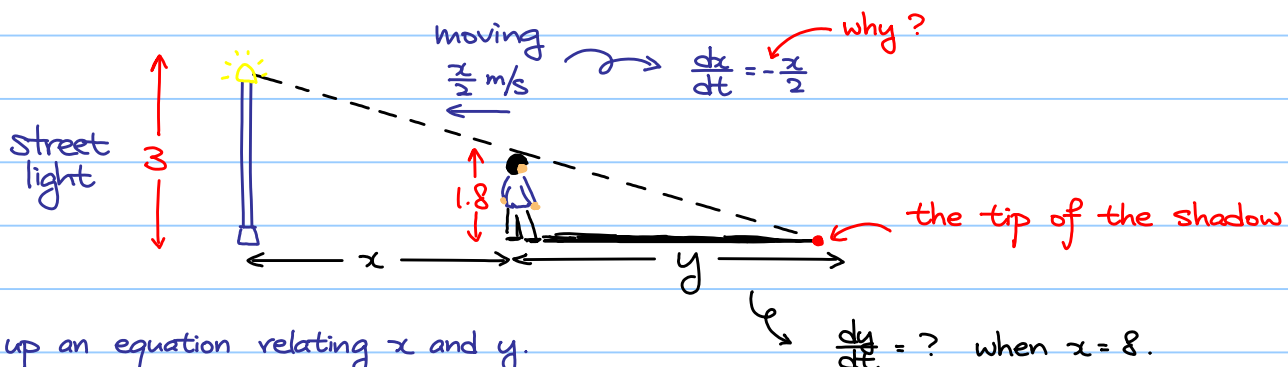
Note: Reasonable !

$\dfrac{dx}{dt} = 1.4 > 0$, i.e. pollutant is increasing.

$\dfrac{dF}{dt} = -70 < 0$, i.e. population of fish is decreasing.

e.g.

speed $= \frac{1}{2} \times$ distance between the street light and him.

street light 3m

1.8m

8m

shadow

rate of change of the shadow when he is 8m away from the street light = ?

moving $\frac{x}{2}$ m/s $\quad\rightsquigarrow\quad \dfrac{dx}{dt} = -\dfrac{x}{2} \quad\leftarrow$ why?

street light 3

1.8

x y

the tip of the shadow

$\dfrac{dy}{dt} = ?$ when $x = 8$.

Setting up an equation relating x and y.

$$\frac{1.8}{x} = \frac{3}{x+y}$$

$$1.8x = 1.2y$$

$$3x = 2y$$

differentiate both sides with respect to t.

$$3\frac{dx}{dt} = 2\frac{dy}{dt}$$

$$3\left(-\frac{x}{2}\right) = 2\frac{dy}{dt}$$

When $x = 8$, $\dfrac{dy}{dt} = -6$.

Furthermore :

What is the speed of the tip of the shadow?
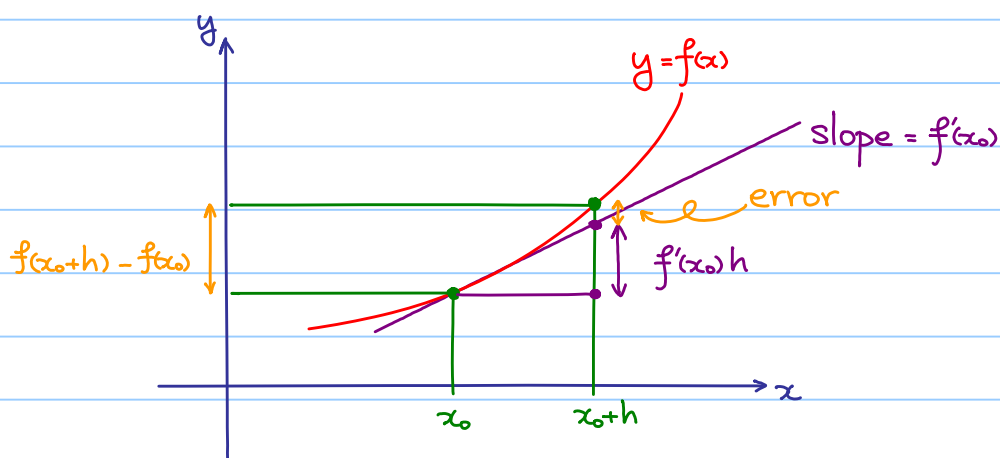
Ans : $\dfrac{d(x+y)}{dt}$!

Marginal Analysis:

Idea: $y = f(x)$

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$\approx \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{if } h > 0, \text{ but } h \text{ is small.}$$

$$f(x_0 + h) - f(x_0) \approx f'(x_0) h$$

e.g. $N(t) =$ GDP of a country, t years after 2015.
$$= t^2 + 4t + 200 \quad \text{(billion dollars)}$$
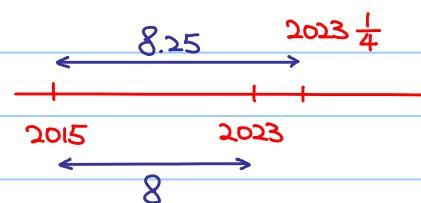
$$N'(t) = 2t + 4$$

change of GDP during the first quarter of 2023
$$= N(8.25) - N(8)$$
$$\approx N'(8) \times 0.25$$
$$= 20 \times 0.25$$
$$= 5$$

$$N(8.25) - N(8) = 5.0625$$

Open Question:

1) Is it a good approximation? Why?

2) Why do we use this approximation?

Indefinite Integral :

Antiderivative : A function $F(x)$ is said to be an antiderivative of $f(x)$ if $F'(x) = f(x)$.

The process of finding antiderivatives is called indefinite integration.

e.g. If $f(x) = 2x$, $F(x) = x^2$,
then we have $F'(x) = f(x)$, so $F(x)$ is an antiderivative of $f(x)$.

However, consider $F(x) = x^2 + C$, where C is a constant.
Then, we still have $F'(x) = f(x)$.
Therefore, antiderivative of a function $f(x)$ is NOT unique.
That is why we call "an" antiderivative instead of "the" antiderivative.

Natural question : If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$,
what is the relation between them ?

Answer : $F(x)$ and $G(x)$ differ by a constant.
proof : Suppose $F'(x) = G'(x) = f(x)$
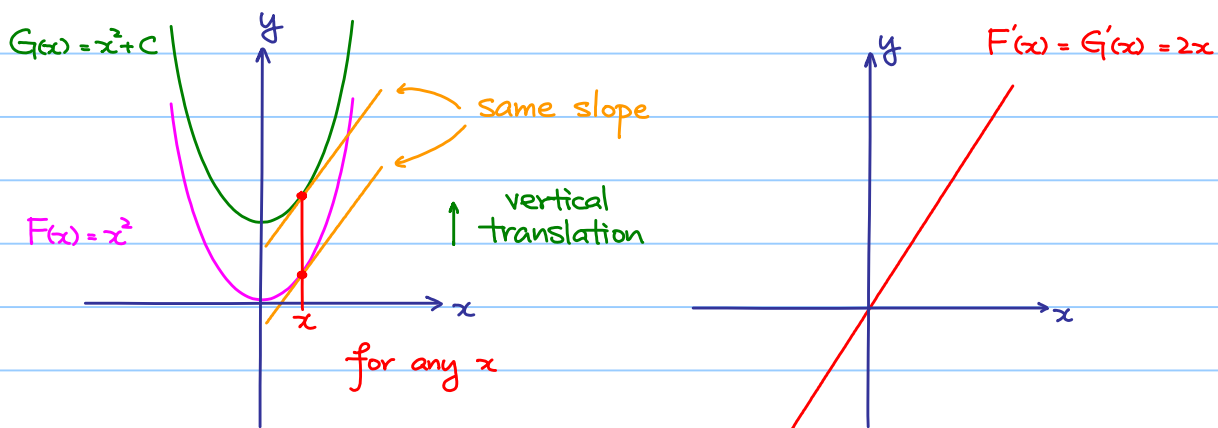Let $H(x) = F(x) - G(x)$
Then $H'(x) = F'(x) - G'(x) = 0$
∴ $H(x)$ is a constant function, i.e. $H(x) = C$ for some constant C.
i.e. $F(x) = G(x) + C$

Therefore, antiderivative of a function $f(x)$ is NOT unique,
but it is unique up to a constant.

e.g. If $f(x) = 2x$, $F(x) = x^2$

then we have $F'(x) = f(x)$, so $F(x) = x^2$ is an antiderivative of $f(x) = 2x$

and all antiderivatives of $f(x)$ must be of the form $x^2 + C$.

$G(x) = x^2 + C$

same slope

vertical translation

$F(x) = x^2$

x

for any x

$F'(x) = G'(x) = 2x$

If $F(x)$ is an antiderivative of $f(x)$, we write

integrand

$$\int f(x) \, dx = F(x) + C$$

integral symbol   variable of integration

e.g. $\int 2x \, dx = x^2 + C$

If $F(x)$ is an antiderivative of $f(x)$,

$F(x)$ $\xrightarrow{\text{differentiate}}$ $f(x)$ $\xrightarrow{\text{integrate}}$ $\int f(x) \, dx = F(x) + C$

= original function up to a constant

Note: When we write $\int f(x) \, dx$, sometimes it may be regarded as a class of functions

Rules for Integrating Common Functions

1) $\int k\,dx = kx + C$, for constant k.

Note: $\frac{d}{dx}(kx+C) = k$

2) $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, for all n except -1.

Note: $\frac{d}{dx}(\frac{1}{n+1}x^{n+1} + C) = x^n$

3) $\int \frac{1}{x}\,dx = \ln|x| + C$ (Interesting when $x<0$)

Note: $\frac{d}{dx}(\ln|x| + C) = \frac{1}{x}$

4) $\int e^x\,dx = e^x + C$

Note: $\frac{d}{dx}(e^x + C) = e^x$