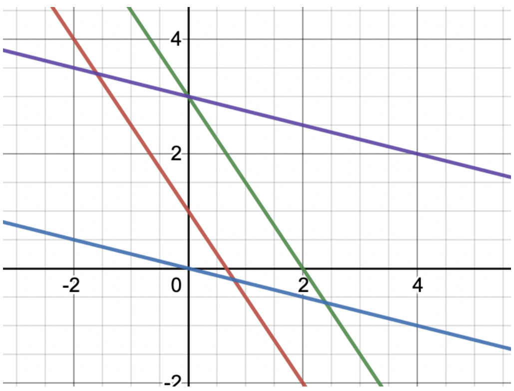
Assignment 7

Q7

7. Use the transformation in Exercise 3 to evaluate the integral

$$\iint_R (3x^2 + 14xy + 8y^2)\, dx\, dy$$

for the region R in the first quadrant bounded by the lines $y = -(3/2)x + 1$, $y = -(3/2)x + 3$, $y = -(1/4)x$, and $y = -(1/4)x + 1$.



Solution. 1.omit the "1st quadrant" condition in the question.
By letting

$$\begin{cases} \text{u=3x+2y} \\ \text{v=x+4y} \end{cases}$$

the region of integral is changed to

$$\begin{cases} 2\leq u \leq 6 \\ 0\leq v \leq 4 \end{cases}$$

The absolute value of the Jacobian of the transformation is $|J(u,v)| = |\frac{\partial(x,y)}{\partial(u,v)}| = \frac{1}{10}$
Function $f = 3x^2 + 14xy + 8y^2 = uv$.
So the integral

$$I = \int_0^4 \int_2^6 \frac{1}{10} uv \; dudv = \frac{64}{5}$$

2. consider the condition "1st quadrant" in the question.
From $x \geq 0$ we have $2u \geq v$, from $y \geq 0$ we have $3v \geq u$
The integral

$$I = \int_2^6 \int_{u/3}^4 \frac{1}{10} uv \; dvdu = \frac{64}{5} - \frac{16}{9}$$

Q12

12. The area of an ellipse The area πab of the ellipse $x^2/a^2 + y^2/b^2 = 1$ can be found by integrating the function $f(x, y) = 1$ over the region bounded by the ellipse in the xy-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au, y = bv$ and evaluate the transformed integral over the disk G: $u^2 + v^2 \leq 1$ in the uv-plane. Find the area this way.

Solution.
To calculate the integral we use generalized polar coordinates by making the following change of variables:

$$x = ar\cos\theta, \quad y = br\sin\theta.$$

The absolute value of the Jacobian of the transformation is $|I| = abr$.
The integral in the new coordinates becomes

$$\begin{aligned} I &= \iint\limits_{U} 1 \; dxdy \\ &= ab \iint\limits_{U'} r \; drd\theta. \end{aligned}$$

The region of integration U' in polar coordinates is a rectangular and defined by the inequalities

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi \quad .$$

Then the area can be written as

$$I = ab \iint\limits_{U'} r \; drd\theta = ab \int\limits_{0}^{2\pi} d\theta \int\limits_{0}^{1} rdr = ab \cdot 2\pi \cdot \frac{1}{2} = ab\pi.$$

Q16

16. Use the transformation $x = u^2 - v^2, y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2} \, dy \, dx.$$

(*Hint:* Show that the image of the triangular region G with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ in the uv-plane is the region of integration R in the xy-plane defined by the limits of integration.)

Solution.
The new region is the above-mentioned triangular.

The absolute value of the Jacobian of the transformation is $|I| = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2+v^2)$.
Function $f = \sqrt{u^4+v^4-2u^2v^2+4u^2v^2} = u^2+v^2$
The integral in the new coordinates becomes

$$\int_0^1 \int_0^u 4(u^2+v^2)^2 \ dvdu = \frac{56}{45}$$

Q20

20. Let D be the region in xyz-space defined by the inequalities

$$1 \leq x \leq 2, \quad 0 \leq xy \leq 2, \quad 0 \leq z \leq 1.$$

Evaluate

$$\iiint_D (x^2y + 3xyz)\, dx\, dy\, dz$$

by applying the transformation

$$u = x, \quad v = xy, \quad w = 3z$$

and integrating over an appropriate region G in uvw-space.

Solution.
the region of integral is changed to

$$\begin{cases} 1 \leq u \leq 2 \\ 0 \leq v \leq 2 \\ 0 \leq w \leq 3 \end{cases}$$

The absolute value of the Jacobian of the transformation is $|I| = \frac{1}{3u}$.
Function $f = uv + vw$
The integral in the new coordinates becomes

$$\frac{1}{3}\iiint_{U'} v + \frac{vw}{u} = 2 + 3ln2$$