Tutorial3

20 October 2021

1 Change of Variables

Let a triple integral be given in the Cartesian coordinates x, y, z in the region U

$$\iiint\limits_U f(x, y, z)\, dxdydz.$$

We need to calculate this integral in the new coordinates u, v, w The relationship between the old and new coordinates is given by

$$x = \varphi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w).$$

The Jacobian of transformation $I(u, v, w)$ equal to

$$I(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix},$$

is non-zero and keeps a constant sign everywhere in the region of integration U. Then the formula for change of variables in triple integrals is written as

$$\iiint\limits_U f(x, y, z)\, dxdydz = \iiint\limits_{U'} f(\varphi, \psi, \chi)\, |I(u, v, w)|\, dudvdw.$$

For calculation simplicity, sometimes we calculate first

$$I^{-1}(u, v, w) = \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

Example 1

Find the volume of the region U defined by the inequalities

$$0 \leq z \leq 2, \; 0 \leq y + z \leq 5, 0 \leq x + y + z \leq 10.$$

Solution. Make the following replacement:

$$u = x + y + z, v = y + z, w = z.$$

The region of integration U' in the new variables u,v,w is defined by the inequalities

$$0 \leq u \leq 10, 0 \leq v \leq 5, 0 \leq w \leq 2.$$

The volume of the solid is

$$V = \iiint\limits_U dxdydz = \iiint\limits_{U'} |I(u, v, w)|\, dudvdw.$$

Find first the Jacobian of the inverse transformation:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1.$$

Then

$$|I(u,v,w)| = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\left(\frac{\partial(u,v,w)}{\partial(x,y,z)}\right)^{-1}\right| = 1.$$

Hence, the volume of the solid is

$$V = \iiint\limits_{U'} |I(u,v,w)|\, du dv dw = \iiint\limits_{U'} du dv dw = \int\limits_0^{10} du \int\limits_0^5 dv \int\limits_0^2 dw = 10 \cdot 5 \cdot 2 = 100.$$

Example 2

Evaluate

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x-y) e^{(2x-y)^2}$$

Solution. Let $t = 2x - y \in [0, 4]$. $(\star)$
Then

$$\frac{\partial(t,y)}{\partial(x,y)} = 2$$

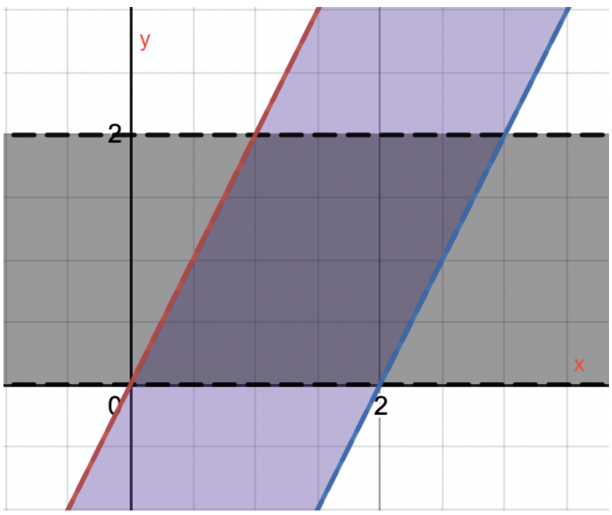
Therefore, the integral is equal to

$$\int_0^2 \int_0^4 y^3 t e^{t^2} \frac{1}{2} dt dy = \frac{1}{2} (\int_0^2 y^3 dy)(\int_0^4 t e^{t^2}) = \frac{1}{2} [\frac{y^4}{4}]_0^2 [\frac{e^{t^2}}{2}]_0^4 = \frac{1}{2} \times 4 \times \frac{e^{16}-1}{2} = e^{16} - 1$$

....

$(\star)$find the range of t.
The region of integral is

$$\frac{y}{2} \leq x \leq \frac{(y+4)}{2}$$
$$0 \leq y \leq 2$$



View t as a constant, then $y = 2x - t$ are a series of lines whose slope is 2 and y-intercept is -t. Then t has its minimum on the red line $t_{min} = 0$, and has its maximum on the blue line $t_{max} = 4$.

Example 3

Find the triple integral

$$\iiint\limits_U \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)dxdydz,$$

where the region is bounded by the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Solution. To calculate the integral we use generalized spherical coordinates by making the following change of variables:

$$x=a\rho\cos\theta\sin\varphi,\;\; y=b\rho\sin\theta\sin\varphi,\;\; z=c\rho\cos\varphi.$$

The absolute value of the Jacobian of the transformation is $|I|=abc\rho^2\sin\varphi$.

The integral in the new coordinates becomes

$$\begin{aligned}
I&=\iiint\limits_U \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)dxdydz\\
&=\iiint\limits_{U'}\left[\frac{(a\rho\cos\theta\sin\varphi)^2}{a^2}+\frac{(b\rho\sin\theta\sin\varphi)^2}{b^2}+\frac{(c\rho\cos\varphi)^2}{c^2}\right]abc\rho^2\sin\varphi d\rho d\theta d\varphi\\
&=\iiint\limits_{U'}\left[\rho^2\cos^2\theta\sin^2\varphi+\rho^2\sin^2\theta\sin^2\varphi+\rho^2\cos^2\varphi\right]abc\rho^2\sin\varphi d\rho d\theta d\varphi\\
&=\iiint\limits_{U'}\left[\rho^2\sin^2\varphi\underbrace{\left(\cos^2\theta+\sin^2\theta\right)}_{1}+\rho^2\cos^2\varphi\right]abc\rho^2\sin\varphi d\rho d\theta d\varphi\\
&=\iiint\limits_{U'}\rho^2\underbrace{\left(\sin^2\varphi+\cos^2\varphi\right)}_{1}\cdot abc\rho^2\sin\varphi d\rho d\theta d\varphi\\
&=abc\iiint\limits_{U'}\rho^4\sin\varphi d\rho d\theta d\varphi.
\end{aligned}$$

The region of integration U' in spherical coordinates is a rectangular parallelepiped and defined by the inequalities

$$0\le\rho\le 1,\;\; 0\le\theta\le 2\pi,\;\; 0\le\varphi\le\pi.$$

Then the triple integral can be written as

$$\begin{aligned}
I&=abc\iiint\limits_{U'}\rho^4\sin\varphi d\rho d\theta d\varphi=abc\int_0^{2\pi}d\theta\int_0^1\rho^4 d\rho\int_0^{\pi}\sin\varphi d\varphi=abc\int_0^{2\pi}d\theta\int_0^1\rho^4 d\rho\cdot\left[(-\cos\varphi)\big|_0^{\pi}\right]\\
&=abc\int_0^{2\pi}d\theta\int_0^1\rho^4 d\rho\cdot(-\cos\pi+\cos 0)=2abc\int_0^{2\pi}d\theta\int_0^1\rho^4 d\rho=2abc\int_0^{2\pi}d\theta\cdot\left[\left(\frac{\rho^5}{5}\right)\Big|_0^1\right]\\
&=\frac{2abc}{5}\int_0^{2\pi}d\theta=\frac{2abc}{5}\cdot\left[\theta\big|_0^{2\pi}\right]=\frac{2abc}{5}\cdot 2\pi=\frac{4abc\pi}{5}.
\end{aligned}$$

2 Applications of Integrals

Areas

Cartesian coordinates.

$$A = \int_a^b \int_{g(x)}^{h(x)} 1 dy dx$$

$$A = \int_c^d \int_{p(y)}^{q(y)} 1 dx dy$$

Polar coordinates.

$$A = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r dr d\theta$$

Volume

Cartesian coordinates.

$$V = \iiint\limits_U dx dy dz$$

Cylindrical coordinates.

$$V = \iiint\limits_U r dr d\theta dz$$

Spherical coordinates

$$V = \iiint\limits_U \rho^2 \sin\theta d\rho d\phi d\theta$$

Mass

(The density at point (x,y,z) is $\delta(x, y, z)$)

$$m = \iiint\limits_U \delta(x, y, z) dx dy dz$$

Static Moments

The static moments of the solid about the coordinate planes Oxy, Oxz, Oyz are given by

$$M_{xy} = \int\limits_U z\delta\left(x, y, z\right) dx dy dz, \quad M_{yz} = \int\limits_U x\delta\left(x, y, z\right) dx dy dz, \quad M_{xz} = \int\limits_U y\delta\left(x, y, z\right) dx dy dz.$$

Center of gravity

The coordinates of the center of gravity of the solid are described by

$$\bar{x} = \frac{M_{yz}}{m} = \frac{\iiint\limits_U x\delta\left(x, y, z\right) dx dy dz}{\iiint\limits_U \delta\left(x, y, z\right) dx dy dz}, \quad \bar{y} = \frac{M_{xz}}{m} = \frac{\iiint\limits_U y\delta\left(x, y, z\right) dx dy dz}{\iiint\limits_U \delta\left(x, y, z\right) dx dy dz}, \quad \bar{z} = \frac{M_{xy}}{m} = \frac{\iiint\limits_U z\delta\left(x, y, z\right) dx dy dz}{\iiint\limits_U \delta\left(x, y, z\right) dx dy dz}.$$

Moments of Inertia

1) The moments of inertia of a solid about the **coordinate planes** Oxy, Oxz, Oyz are given by

$$I_{xy} = \iiint\limits_U z^2 \delta(x,y,z)\, dxdydz, \quad I_{yz} = \iiint\limits_U x^2 \delta(x,y,z)\, dxdydz, \quad I_{xz} = \iiint\limits_U y^2 \delta(x,y,z)\, dxdydz,$$

2) The moments of inertia of a solid about the **coordinate axes** Ox, Oy, Oz are expressed by the formulas

$$I_x = \iiint\limits_U \left(y^2+z^2\right) \delta(x,y,z)\, dxdydz, \quad I_y = \iiint\limits_U \left(x^2+z^2\right) \delta(x,y,z)\, dxdydz, \quad I_z = \iiint\limits_U \left(x^2+y^2\right) \delta(x,y,z)\, dxdydz.$$

3) The moments of inertia of a solid about **the axes** L are expressed by the formulas

$$I_L = \iiint\limits_U r^2 \delta(x,y,z)\, dxdydz.$$

$r(x,y,z)$ is the distance from point (x,y,z) to line L.
4) The moment of inertia about the **origin** is

$$I_0 = \iiint\limits_U \left(x^2+y^2+z^2\right) \delta(x,y,z)\, dxdydz.$$

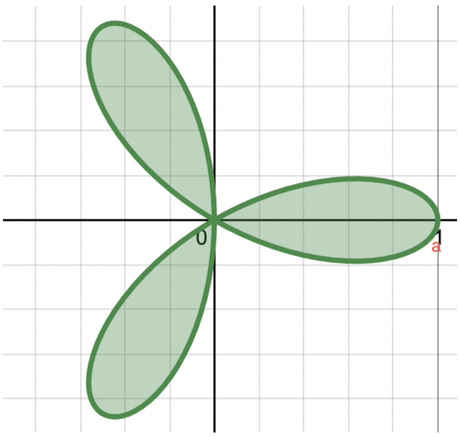
Example 4

Find the area of $r = a\cos 3\theta$, $a > 0$. Can you express the curve in Cartesian coordinates?

This rose has three leaves, one of which lies over $\theta \in [\pi/6, \pi/6]$. The area of one leaf is given by

$$A_1 = \int_{-\pi/6}^{\pi/6} \int_0^{a\cos 3\theta} r dr d\theta = \frac{a^2}{2} \int_{-\pi/6}^{\pi/6} \cos^2 3\theta d\theta = \frac{a^2}{2} \int_{-\pi/6}^{\pi/6} \frac{\cos 6\theta + 1}{2} d\theta = \frac{\pi a^2}{12}$$

Total area $A = 3 \times A_1 = \frac{\pi a^2}{4}$



Using $\cos 3\theta = 4\cos 3\theta 3\cos\theta$, the curve $r = a\cos 3\theta$ becomes

$$r = a(4\frac{x^3}{r^3} - 3\frac{x}{r})$$

$$(x^2+y^2)^2 = a(4x^3 - 3x(x^2+y^2)) = a(x^3 - 3xy^2)$$