Supplementary concepts in Metric spaces (Relevant to Function Spaces)

(A) Separable Spaces

Def: A metric space (X, d) is said to be <u>separable</u> if there is a <u>countable</u> <u>dense</u> subset $E \subset X$.

(i.e. E is countable and $\overline{E} = X$)

egs: • $\mathbb{R}$ is <u>separable</u> as $\mathbb{Q}$ is countable and $\overline{\mathbb{Q}} = \mathbb{R}$.

• $\mathbb{R}^n$ is separable (see the proof of the Ascoli's Theorem)

• $(C[a,b], d_\infty)$ is <u>separable</u> by Weierstrass Approximation Theorem: $\mathcal{P}$ = set of polynomials on $[a,b]$. Then $\overline{\mathcal{P}} = C[a,b]$.

$\overline{\mathbb{Q}} = \mathbb{R} \Rightarrow$ polynomials approximated by polynomials with <u>rational</u> coefficients in supnorm.

Hence $\quad \overline{\mathcal{P}_{\mathbb{Q}}} = \overline{\mathcal{P}} = C[a,b]$, where

$\mathcal{P}_{\mathbb{Q}}$ = set of polynomials with rational coefficients is countable.

e.g. But $M[a,b]$ = space of bounded functions on $[a,b]$ is nonseparable (omitted)

(B) Compactness
(Recall in the §4.1 Ascoli's Thm)

Def: A set K in a metric space is said to be compact if every sequence in K has a subsequence converging to a point in K.

(i.e. K is precompact and closed)
(i.e. K has Bolzano-Weierstrass property)

Thm (X, d) = metric space
$K \subset X$ compact.

Then (1) K is closed.

(2) K is bounded.

(3) K is complete.

(4) K is separable.

Thm (X, d) = metric space.

Then

K is compact (ie Bolzano-Weirstrass Property)

$\Leftrightarrow$ Every open cover of K has a finite subcover (Heine-Borel Property)

(c) Total Boundedness

Def Let (X, d) be a metric space.
$S \subset X$ is called totally bounded
if $\forall \varepsilon > 0$, $\exists$ finite set $\{x_1, \cdots, x_n\} \subset X$
s.t. $S \subset \bigcup_{i=1}^{n} B_\varepsilon(x_i)$.

Thm A metric space X is totally bounded
$\Leftrightarrow$ every sequence has a Cauchy subsequence

(X may not complete, Cauchy subseq. may not converge)

Thm A metric space is compact $\Leftrightarrow$
it is complete and totally bounded.

Final Exam:

Ch1 Fourier Series

- Riemann-Lebesgue Lemma
- pointwise and uniform convergence
- Weierstrass Approximation Theorem
- L^2-convergence (mean convergence)
- Parserval's Identity

Ch2 Metric Spaces

- Basic notations
- Open and Closed Sets
- Interior, closure & boundary
- Elementary Inequalities for Functions
  (Young's, Hölder's, Minkowski's)

Ch3 Contraction Mapping Principle

- Completeness
- Fixed points & Contraction
- Perturbation of Identity
- Inverse Function Theorem (Implicit Function Thm)
- Picard-Lindelöf Thm (IVP in ODE)

Ch4 Space of Continuous Functions

- Ascoli's Theorem
  (equicontinuity, uniform bddness, precompact)
- Arzela's Theorem
- Cauchy-Peano Thm (IVP in ODE)
- Baire Category Thm
  (nowhere dense, 1st category, residual)
- Applications of Baire Category Thm
  (eg nowhere differentiable continuous functions & etc)