Note: E is nowhere dense $\Leftrightarrow$ $X \setminus \overline{E}$ is dense in X.

Pf: E is nowhere dense

$\Leftrightarrow$ $\forall x \in X$ and any $r > 0$, $B_r(x) \not\subset \overline{E}$.

$\Leftrightarrow$ $\forall x \in X$ and any $r > 0$, $B_r(x) \cap (X \setminus \overline{E}) \neq \phi$

$\Leftrightarrow$ $X \setminus \overline{E}$ is dense. #

Def: Let (X, d) be a metric space. A point $x \in X$ is called an isolated point if $\{x\}$ is open in X.

Note: As $\{x\}$ is always closed in a metric space, $\{x\}$ is both open and closed in X iff x is an isolated point.

eg: • $\mathbb{R}$ has no isolated points.

• All points in $\mathbb{Z}$ (subspace of $\mathbb{R}$) are isolated. ($\forall n \in \mathbb{Z}$, $\{n\} = B_{\frac{1}{2}}(n)$ is open in $\mathbb{Z}$)

Prop 4.7 Let (X, d) be a metric space.

(a) E is nowhere dense in X $\Rightarrow$

$\bar{E}$ is nowhere dense in X and E' is nowhere dense in X if $E' \subset E$.

(b) The union of finitely many nowhere dense sets (in X) is nowhere dense (in X).

(c) If (X, d) has no isolated point, then every finite set is nowhere dense.

Pf: (a) Trivial.

(b) Let E_1, E_2 be nowhere dense sets.

Then $G_1 = X \setminus \bar{E}_1$ and $G_2 = X \setminus \bar{E}_2$ are open dense set. Clearly $G_1 \cap G_2$ is open. We claim that it is dense in X.

In fact, $\forall\, x \in X$ and $r > 0$.

G_1 dense $\Rightarrow B_r(x) \cap G_1 \neq \phi$

$\Rightarrow \exists\, x_1 \in B_r(x) \cap G_1$

Since $B_r(x) \cap G_1$ is open, $\exists\, \rho > 0$ such that

$B_\rho(x_1) \subset B_r(x) \cap G_1$.

Now G_2 dense $\Rightarrow B_\rho(x_1) \cap G_2 \neq \phi$.

$\Rightarrow \quad B_r(x) \cap G_1 \cap G_2 \neq \phi$

This proves the claim and hence

$\bar{E}_1 \cup \bar{E}_2 = X \setminus (G_1 \cap G_2)$ is nowhere dense

By (a), $E_1 \cup E_2$ is also nowhere dense.

Then introduction $\Rightarrow \bigcup_{i=1}^{k} E_i$ is nowhere dense provided $E_1, \cdots E_k$ are nowhere dense.

(c) If (X, d) has no isolated point, then $\forall x \in X$, $\{x\}$ is nowhere dense in X.

Otherwise $\{x\} = \overline{\{x\}}$ contains some open balls $B_r(y)$. This implies $y = x$ and $\{x\} = B_r(x)$ is open, contradicting the assumption that x is not isolated.

Then by part (b), any finite set is nowhere dense. #

eg: $(\mathbb{R}, d(x,y) = |x-y|)$ has no isolated point

$\Rightarrow$ any $\{x_1, \cdots, x_n\}$ is nowhere dense.

But for countable subsets, we have no such conclusion:

- $\mathbb{N} = \{1, 2, 3, \cdots\}$ countable and nowhere dense.
- $\mathbb{Q}$ countable, but not nowhere dense.

Examples in infinite dimensional normed spaces

(Ref: "Elementary Real Analysis" by B. Thomson, J. Bruckner & A. Brucker)

eg Let $M[a,b]$ = space of bounded functions on $[a,b]$.

Then $\|f\|_\infty = \sup_{[a,b]} |f(x)|$ is well-defined and is a norm on $M[a,b]$.

Clearly $(C[a,b], d_\infty)$ is a metric (also vector) subspace of $(M[a,b], d_\infty)$.

<u>Claim</u>: $C[a,b]$ is nowhere dense in $M[a,b]$ (with respect to d_∞ metric).

(1) Clearly, $C[a,b]$ is closed (uniform limit of cts fcns is cts)

We only need to show that

(2) $\forall B_\varepsilon^\infty(f) \subset M[a,b]$, $B_\varepsilon^\infty(f) \cap (M[a,b] \setminus C[a,b]) \neq \phi$.

(i) If $f \in M[a,b] \setminus C[a,b]$, we are done.

(ii) If $f \in C[a,b]$,

define $g(x) = \begin{cases} f(x) + \frac{\varepsilon}{2}, & x \in [a,b] \cap \mathbb{Q} \\ f(x) - \frac{\varepsilon}{2}, & x \in [a,b] \setminus \mathbb{Q}. \end{cases}$

Then $g(x) - f(x) = \pm \frac{\varepsilon}{2}$

$\Rightarrow \quad \|g - f\|_\infty = \frac{\varepsilon}{2}$

$\therefore \quad g \in B_\varepsilon^\infty(f)$.

Since $[a,b] \cap \mathbb{Q}$ and $[a,b] \setminus \mathbb{Q}$ are dense in $[a,b]$,

We have

$$\limsup_{x \to a} g(x) = f(a) + \frac{\varepsilon}{2} \quad \&$$

$$\liminf_{x \to a} g(x) = f(a) - \frac{\varepsilon}{2}$$

(We've used that $f \in C[a,b]$.)

$\Rightarrow \quad g \in M[a,b] \setminus C[a,b]$.

$\therefore \quad B_\varepsilon^\infty(f) \cap (M[a,b] \setminus C[a,b]) \neq \emptyset$.

$\#$

(Or by contradiction: if $g \in C[a,b]$, then $g - f = \begin{cases} \frac{\varepsilon}{2}, & [a,b] \cap \mathbb{Q} \\ -\frac{\varepsilon}{2}, & [a,b] \setminus \mathbb{Q} \end{cases}$

is continuous, which is impossible.

Hence $g \in M[a,b] \setminus C[a,b]$.)

eg. Let l_∞ = space of bounded sequences with d_∞ metric.

Let $\mathcal{C}$ = subset of convergent sequences.

Then $\mathcal{C}$ is nowhere dense in (l_∞, d_∞).

Pf: We only need to show (1) & (2) in the following:

(1) $\mathcal{C}$ is closed.

Pf of (1) Let $x = \{x_n\} \in l_\infty \setminus \mathcal{C}$.

Then x_n diverges and

$$L = \limsup_n x_n > \liminf_n x_n = l.$$

Take $\varepsilon = \frac{L-l}{3} > 0$

then $\forall\, y = \{y_n\} \in B^\infty_\varepsilon(x)$, we have

$$x_n - \varepsilon < y_n < x_n + \varepsilon \quad \forall n$$

$$\Rightarrow \begin{cases} \limsup x_n - \varepsilon \le \limsup y_n \\ \liminf y_n \le \liminf x_n + \varepsilon \end{cases}$$

$$\Rightarrow \limsup y_n \ge L - \varepsilon = \frac{2L+l}{3} > \frac{L+2l}{3} \quad (\text{since } L > l)$$

$$= l + \varepsilon \ge \liminf y_n$$

$\Rightarrow y = \{y_n\}$ is divergent.

Hence $B^\infty_\varepsilon(x) \subset l_\infty \setminus \mathcal{C}$ & This proves (1).

(2) $l_\infty \setminus \mathcal{C}$ is dense.

Pf of (2): Let $B^\infty_\varepsilon(x)$ be a ball in l_∞.

We need to show that $B^\infty_\varepsilon(x) \cap (l_\infty \setminus \mathcal{C}) \neq \emptyset$.

If $x \in l_\infty \setminus \mathcal{C}$, we are done.

If $x \in \mathcal{C}$, then $x = \{x_n\}$ is convergent.

Let $L = \lim_{n\to\infty} x_n$.

Then $\exists\ n_0 > 0$ s.t. $|x_n - L| < \frac{\varepsilon}{3}$, $\forall\ n \geq n_0$.

Define $y = \{y_n\} \in l_\infty$ by

$$y_n = \begin{cases} x_n, & \text{if } n < n_0 \\ L + \frac{\varepsilon}{3}, & \text{if } n \geq n_0 \ \& \ n \text{ odd} \\ L - \frac{\varepsilon}{3}, & \text{if } n \geq n_0 \ \& \ n \text{ even.} \end{cases}$$

Then $|x_n - y_n| = 0$ if $n < n_0$ and

$$|x_n - y_n| \leq |x_n - L| + |L - y_n| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}, \ \forall\ n \geq n_0$$

$\Rightarrow\ d_\infty(x,y) \leq \frac{2\varepsilon}{3} < \varepsilon$. i.e. $y \in B^\infty_\varepsilon(x)$.

However $\limsup y_n = L + \frac{\varepsilon}{3} > L - \frac{\varepsilon}{3} = \liminf y_n$.

$\therefore\ y \in l_\infty \setminus \mathcal{C}$.

$\Rightarrow\ B^\infty_\varepsilon(x) \cap (l_\infty \setminus \mathcal{C}) \neq \emptyset$. ※

Def: • A set in a metric space is called of first category (or meager) if it can be expressed as a countable union of nowhere dense sets.

- A set is of second category if it is not of first category.
- A set is called residual if its complement is of first category.

Prop 4.8 Let (X, d) be a metric space.

(a) Every subset of a set of first category is of first category.

(b) The union of countable many sets of first category is of first category.

(c) If (X, d) has no isolated point, then every countable subset of X is of first category.

Pf: (a) Let $E \subset X$ be a set of 1^{st} category.

Then $E = \bigcup_{n=1}^{\infty} E_n$ for some nowhere dense sets

E_n, $n = 1, 2, 3, \cdots$

Let $F \subset E$, then by Prop 4.7 (a),

$F \cap E_n$ is nowhere dense, $\forall n$.

Hence $F = F \cap E = \bigcup_{n=1}^{\infty} (F \cap E_n)$ is of 1^{st} category

#

(b) Let $E_n = \bigcup_{k=1}^{\infty} E_{n,k}$, $E_{n,k}$ = nowhere dense.

$$\Rightarrow \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{n,k}$$

$$= \bigcup_{(n,k) \in \mathbb{N} \times \mathbb{N}} E_{n,k} \text{ is of } 1^{st} \text{ category.}$$

#

(c) If $E = \{x_i\}_{i=1}^{\infty} \subset X$, then prop 4.7 (c)

$\Rightarrow$ $\{x_i\}$ is nowhere dense $\forall i$

$\Rightarrow$ $E = \bigcup_{i=1}^{\infty} \{x_i\}$ is of 1^{st} category. #

Prop 4.8' Let (X, d) be a metric space.

(a) Every subset containing a residual set is residual.

(b) The intersection of countable many residual sets is a residual set.

(c) If (X, d) has no isolated point, then complement of a countable set is a residual set.

(Pf: By taking complement in Prop 4.8.)

eg 4.5: $\mathbb{R}$ has no isolated point (in standard metric)

$\Rightarrow$ $\{q_j\}$ is nowhere dense for any rational number

$\Rightarrow$ $\mathbb{Q}$ is of 1^{st} category since it is a countable union of $\{q_j\}$.

Hence $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ the set of irrational numbers is a residual set in $\mathbb{R}$.

Thm 4.9 (Baire Category Theorem)

In a complete metric space, any set of 1^{st} category has empty interior.

Pf: Let the complete metric space be (X, d). And let $E = \bigcup_{n=1}^{\infty} E_n \subset X$ be of 1^{st} category,

where E_n is nowhere dense, $\forall n$.

Consider any open metric ball $B_{r_0}(x_0)$ of X.

Since $\overline{E_1}$ has empty interior (by definition of nowhere dense), $(X \setminus \overline{E_1}) \cap B_{r_0}(x_0) \neq \emptyset$.

Let $x_1 \in (X \setminus \overline{E_1}) \cap B_{r_0}(x_0)$

Since both $X \setminus \overline{E_1}$ & $B_{r_0}(x_0)$ are open,

$\exists\ r_1 > 0$ s.t. $\overline{B_{r_1}(x_1)} \subset (X \setminus \overline{E_1}) \cap B_{r_0}(x_0)$.

and $r_1 \leq \frac{r_0}{2}$. (as we can always choose a smaller ball)

Now E_2 is nowhere dense, $\overline{E_2}$ has empty interior.

$\therefore \quad (X \setminus \overline{E_2}) \cap B_{r_1}(x_1) \neq \emptyset$.

Similar to the above, $\exists\ x_2 \in (X \setminus \overline{E_2}) \cap B_{r_1}(x_1)$

and $r_2 > 0$ with $r_2 \leq \frac{r_1}{2}$ s.t.

$$\overline{B_{r_2}(x_2)} \subset (X \setminus \overline{E_2}) \cap B_{r_1}(x_1).$$

Note that $\overline{B_{r_2}(x_2)} \subset (X \setminus \overline{E_2}) \cap B_{r_1}(x_1) \subset (X \setminus \overline{E_1}) \cap B_{r_0}(x_0)$.

Repeating the process, we obtain $\{x_n\}_{n=1}^{\infty} \subset X$ and $\{r_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$ such that

(a) $\overline{B_{r_{n+1}}(x_{n+1})} \subset B_{r_n}(x_n)$,

(b) $r_{n+1} \leq \frac{r_n}{2}$,

(c) $\overline{B_{r_n}(x_n)} \cap \overline{E_j} = \emptyset$, $\forall j = 1, 2, \cdots, n$.

By (a) & (b), $\{x_n\}$ is a Cauchy sequence. Hence completeness of $X \Rightarrow \exists\ x \in X$ s.t.

$$x_n \to x.$$

By (a) again, $x_{n+m} \in \overline{B_{r_n}(x_n)} \quad \forall m = 1, 2, 3, \cdots$

$\Rightarrow \quad x \in \overline{B_{r_n}(x_n)}$.

$\overset{\text{by (4) \& (6)}}{\Rightarrow} \quad x \in X \setminus \overline{E_n}$, and $x \in B_{r_0}(x_0)$

Since n is arbitrary

$$x \in \bigcap_{n=1}^{\infty} (X \setminus \overline{E_n}) = X \setminus \left(\bigcup_{n=1}^{\infty} \overline{E_n} \right)$$

Hence $\left(X \setminus \bigcup_{n=1}^{\infty} \overline{E_n} \right) \cap B_{r_0}(x_0) \neq \emptyset$.

$$\Rightarrow \left(X \setminus \bigcup_{n=1}^{\infty} E_n \right) \cap B_{r_0}(x_0) \supseteq \left(\left(X \setminus \bigcup_{n=1}^{\infty} \overline{E_n} \right) \cap B_{r_0}(x_0) \right) \neq \emptyset.$$

Since $B_{r_0}(x_0)$ is arbitrary, $E = \bigcup_{n=1}^{\infty} E_n$ has empty interior. ※

Recall that E is a closed nowhere dense set

$\Leftrightarrow \quad X \setminus E$ is an open dense set.

Hence Thm 4.9 can be rephrased as

Thm 4.9′ (Baire Category Theorem)

In a complete metric space, countable intersection of open dense sets is dense.

i.e. If (X, d) is complete and $G_n \subset X$ is a sequence of open dense sets in X, then $\bigcap_{n=1}^{\infty} G_n$ is dense.