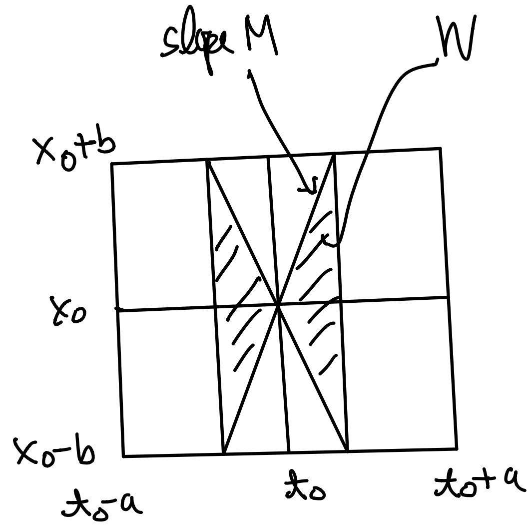
Another approach to Cauchy-Peano Theorem using Ascoli's Theorem

(Piecewise Linear Approximation)

Let $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$

$$M = \sup_R |f(t,x)| \text{ as before.}$$

(May assume $M \geq 1$ as we only need an upper bd)



Define

$$W = \{(t,x) \in R : |x - x_0| \leq M|t - t_0|\}$$

By symmetry, proj(W) onto t-axis is $[t_0 - a', t_0 + a']$ for some $a' \in (0, a]$.

Note that $f \in C(R) \Rightarrow f \in C(W)$

$\Rightarrow$ f is uniformly continuous on W (since W is closed & bounded)

$\Rightarrow$ $\forall \varepsilon > 0, \exists \delta > 0$ such that

$\forall (t_1, x_1), (t_2, x_2) \in W$ with

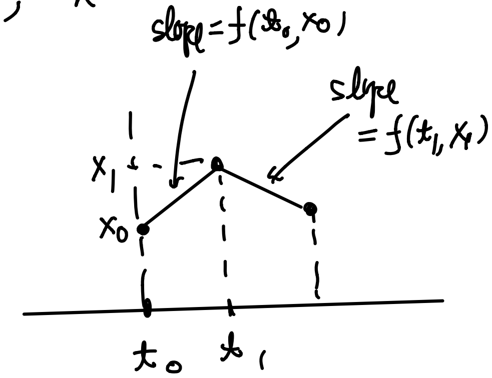
$|t_1 - t_2| < \delta$ and $|x_1 - x_2| < \delta$,

we have $|f(t_2, x_2) - f(t_1, x_1)| < \varepsilon$.

On the (half) interval $[t_0, t_0+a']$,

Choose $\quad t_0 < t_1 < t_2 < \cdots < t_k = t_0 + a'$

with $\quad |t_i - t_{i-1}| < \frac{\delta}{M}$ for $i = 1, \cdots k$



Define a function $k_\varepsilon(t)$ on $[t_0, t_0+a']$

(1) $k_\varepsilon(t_0) = x_0$,

(2) $k_\varepsilon\big|_{[t_{i-1}, t_i]}$ is linear with slope $f(t_{i-1}, x_{i-1})$

where x_i can be determined succesively by :

(i) x_1 detemied by $k_\varepsilon\big|_{[t_0, t_1]}$ is linear passing through (t_0, x_0) and with slope $f(t_0, x_0)$.

(ii) Note that $|f(t_0, x_0)| \leq M$,

$$|x_1 - x_0| \leq M |t_1 - t_0|$$

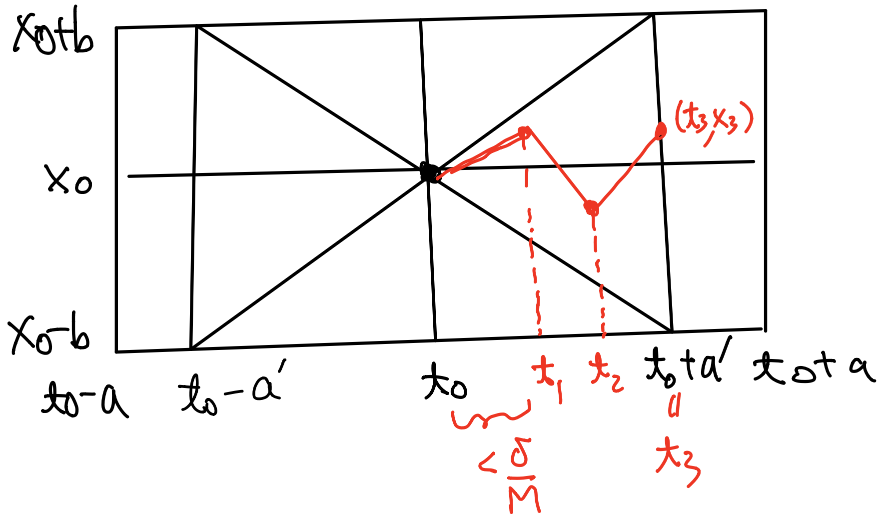
$\therefore$ $(t_1, x_1) \in W \subset R$ and hence $f(t_1, x_1)$ well-defined.

(iii) then x_2 detemined by $k_\varepsilon\big|_{[t_1, t_2]}$ is linear passing through (t_1, x_1) and with slope $f(t_1, x_1)$.

(iv) Similarly, $|f(t_0, x_0)|$ & $|f(t_1, x_1)| \leq M$, we have

$$|x_2 - x_0| \leq M |t_2 - t_0|$$

$\therefore$ $(t_2, x_2) \in W \subset R$ and $f(t_2, x_2)$ well-defined.



graph of $k_\varepsilon(t)$

And so on, the function $k_\varepsilon(t)$ is defined on $[t_0, t_0+a']$

Note that

(1) k_ε is piecewise linear,

(2) $|k_\varepsilon(t) - k_\varepsilon(s)| \leq M\,|t-s|$, $\forall\, t,s \in [t_0, t_0+a']$

(By slopes $|f(t_i, x_i)| \leq M$ on each subinterval.)

$\therefore \{k_\varepsilon\}$ is equicontinuous (as subset of $C[t_0, t_0+a']$)

$\{k_\varepsilon\}$ is also uniformly bounded $[t_0, t_0+a']$.

In fact, W is convex and the ends points (t_i, x_i) with $x_i = k_\varepsilon(t_i)$ belongs to W, we have $(t, k_\varepsilon(t)) \in W$ by piecewise linearity. As $W \subset R$, $|k_\varepsilon(t) - x_0| \leq b$ and hence $|k_\varepsilon(t)| \leq |x_0| + b$, $\forall\, t \in [t_0, t_0+a']$ and $\forall\, \varepsilon > 0$.

Hence Ascoli's Theorem implies that $\{k_\varepsilon\}$ is precompact.

In particular, the sequence $\{k_{\frac{1}{n}}\}_{n=1}^{\infty}$ has a convergent subsequence $\{k_{\frac{1}{n_l}}\}$ in $C[t_0, t_0+a']$ with $k_{\frac{1}{n_l}}(t) \to k(t) \in C[t_0, t_0+a']$, as $l \to +\infty$.

To show $k(t)$ satisfies the different equation, we first show that k_ε is an approximated solution for any $\varepsilon > 0$.

Consider $t \in [t_0, t_0+a']$ and $t \neq t_i$, $i=0,1,\cdots,k-1$

Then $\exists\, j=1,2,\cdots k$ such that (For $\varepsilon = \frac{1}{n_l} > 0$ & corresponding $\delta > 0$)

$$t_{j-1} < t < t_j .$$

Using $|t-t_{j-1}| < |t_j - t_{j-1}| < \frac{\delta}{M}$, we have

$$|k_\varepsilon(t) - k_\varepsilon(t_{j-1})| \leq M|t-t_{j-1}| < \delta,$$

Hence

$$|f(t_{j-1}, k_\varepsilon(t_{j-1})) - f(t, k_\varepsilon(t))| < \varepsilon$$

Since k_ε is piecewise linear,

$$k_\varepsilon'(t) = f(t_{j-1}, k_\varepsilon(t_{j-1})) \quad \text{(by our construction)}$$

Hence

$$| k_\varepsilon'(t) - f(t, k_\varepsilon(t)) | < \varepsilon, \quad \forall t \in [t_0, t_0+a'] \setminus \{t_0, t_1, \cdots, t_k\}.$$

As $k_\varepsilon(t_0) = x_0$, $k_\varepsilon(t)$ is an approximated solution to

$$(\text{IVP}) \quad \begin{cases} \dfrac{dx}{dt} = f(t, x) & \text{on } [t_0, t_0+a'] \\ x(t_0) = x_0 \end{cases}$$

in the sense that
$$\begin{cases} \dfrac{dk_\varepsilon}{dt} = f(t, k_\varepsilon) + \text{remainder} \\ x(t_0) = x_0 \end{cases}$$

with $\|\text{remainder}\|_\infty < \varepsilon$.

Integrating the ODE, we have

$$\Rightarrow \quad k_\varepsilon(t) = k_\varepsilon(t_0) + \sum_{i=1}^{j-1} \int_{t_{i-1}}^{t_i} k_\varepsilon'(s)\,ds + \int_{t_{j-1}}^{t} k_\varepsilon'(s)\,ds$$

$$= x_0 + \int_{t_0}^{t} k_\varepsilon'(s)\,ds$$

$$\Rightarrow \quad \left| k_\varepsilon(t) - x_0 - \int_{t_0}^{t} f(s, k_\varepsilon(s))\,ds \right| \le \int_{t_0}^{t} |k_\varepsilon'(s) - f(s, k_\varepsilon(s))|\,ds < \varepsilon a'.$$

In particular, if we denote $g_\ell = k_{\frac{1}{n_\ell}}$, (ie $\varepsilon = \frac{1}{n_\ell} \to 0$),

then
$$\left| g_\ell(t) - x_0 - \int_{t_0}^{t} f(s, g_\ell(s))ds \right| \leq \frac{a'}{n_\ell}, \quad \forall \ell = 1, 2, 3, \cdots$$

Hence
$$\left| k(t) - x_0 - \int_{t_0}^{t} f(s, k(s))ds \right|$$
$$\leq \left| k(t) - x_0 - \int_{t_0}^{t} f(s, k(s))ds - g_\ell(t) + x_0 + \int_{t_0}^{t} f(s, g_\ell(s))ds \right|$$
$$+ \left| g_\ell(t) - x_0 - \int_{t_0}^{t} f(s, g_\ell(s))ds \right|$$
$$\leq \| k - g_j \|_\infty + \int_{t_0}^{t} |f(s, g_\ell(s)) - f(s, k(s))| ds + \frac{a'}{n_\ell}.$$

Since $\|g_j - k\|_\infty \to 0$ and f is uniform continuity,
$$\int_{t_0}^{t} |f(s, g_j(s)) - f(s, k(s))| ds \to 0 \quad \text{as } j \to +\infty.$$

Therefore by letting $j \to +\infty$, we have
$$k(t) = x_0 + \int_{t_0}^{t} f(s, k(s))ds, \quad \forall t \in [t_0, t_0 + a'].$$

$$\Rightarrow \begin{cases} \dfrac{dk}{dt} = f(t, k(t)) & \forall t \in [t_0, t_0 + a'] \\ k(t_0) = x_0. \end{cases}$$

Similarly argument $\Rightarrow$ $\exists \tilde{k}$ on $t \in [t_0 - a', t_0]$

satisfying $\begin{cases} \dfrac{d\tilde{k}}{dt} = f(t, \tilde{k}(t)) & \forall\, t \in [t_0 - a', t_0] \\ \tilde{k}(t_0) = x_0 . \end{cases}$

Note that by construction

$$\frac{dk}{dt}(t_0) = f(t_0, x_0) = \frac{d\tilde{k}}{dt}(t_0) .$$

Hence
$$x(t) = \begin{cases} k(t) , & t \in [t_0, t_0 + a'] \\ \tilde{k}(t) , & t \in [t_0 - a', t_0] \end{cases}$$

is $C^1[t_0 - a', t_0 + a']$ and solve the (IVP).

$\#\#$

Remarks (i) This proof doesn't need the Picard-Lindelöf Theorem.

(ii) The spirit of this proof is more in line with solving the (IVP) numerically.

(iii) The 1st proof solve "approximated problems"; the 2nd proof solve the (original) problem "approximately".

§4.2 Baire Category Theorem

Def: Let (X, d) be a metric space. A set E in X is dense if $\forall x \in X$, and $\varepsilon > 0$,

$$B_\varepsilon(x) \cap E \neq \emptyset.$$

Notes: (i) Easy to see that E is dense $\Leftrightarrow \overline{E} = X$.

(ii) X is dense (in (X, d)).

eg: If $(X,$ discrete metric$)$, then for $0 < \varepsilon < 1$ & $x \in X$, $B_\varepsilon(x) = \{x\}$. Therefore E is dense in X implies $E = X$. (i.e. X is the only dense set in $(X,$ discrete$)$.)

eg1: In $(\mathbb{R},$ standard metric$)$, $\mathbb{Q}$ & $\mathbb{R} \setminus \mathbb{Q} = \mathbb{I}$ are dense.

eg 2: Weierstrass approximation theorem implies the set of all polynomials $\mathcal{P}$ forms a dense set in $(C[0,1], d_\infty)$.

Def: Let (X, d) be a metric space. A subset $E \subset X$ is called nowhere dense if its closure does not contain any metric ball. (i.e. $\overline{E}$ has empty interior.)

eg: $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ nowhere dense in $\mathbb{R}$. However,

$\mathbb{Q}$ has empty interior but $\overline{\mathbb{Q}} = \mathbb{R}$ has nonempty interior, so $\mathbb{Q}$ is not nowhere dense.