Eg 4.1 (Equicontinuous, but unbounded)

let $X = [-1, 1]$ and consider

$\mathcal{E} = \{ x \in C[-1,1] : x'(t) = t,\ t \in [-1,1] \}$.

$\forall\, x \in \mathcal{E}$, $|x(t) - x(s)| \leq \|x'\|_\infty |t-s| \leq |t-s|$.

$\Rightarrow$ $\mathcal{E}$ is equicontinuous (as above).

But $\mathcal{E}$ is unbounded:

$$x_n(t) = \frac{t^2}{2} + n \in \mathcal{E}.$$

$$\|x_n\|_\infty = \frac{1}{2} + n \to +\infty \text{ as } n \to +\infty.$$

(Clearly, $\{x_n\}$ has no convergent subsequence.)

eg 4.5 (Closed & Bounded, but not Equicontinuous)

Let $\mathcal{B} = \{ f \in C[0,1] : |f(x)| \leq 1,\ \forall\, x \in [0,1] \}$ $\left(\overset{\shortparallel}{\overline{B_1^\infty(0)}} \right)$

Then $\mathcal{B}$ is closed and bounded.

To show that $\mathcal{B}$ is not equicontinuous, we only need to find a subset of $\mathcal{B}$ which is not equicontinuous.

Let $\{f_n(x) = \sin nx\}_{n=1}^{\infty} \subset \mathcal{B}$.

Suppose on the contrary that

$\{f_n(x) = \sin nx\}_{n=1}^{\infty}$ is equicontinuous.

Then for $\varepsilon = \frac{1}{2}$, $\exists\, \delta > 0$ such that

$\forall\, n \geq 1$, & $x, y \in [0,1]$ with $|x-y| < \delta$, we have

$$|\sin nx - \sin ny| < \frac{1}{2}.$$

However, for any $\delta > 0$, if $n > \max\{\frac{\pi}{2\delta}, \frac{\pi}{2}\}$,

we have $x = 0$ & $y = \frac{\pi}{2n} \in [0,1]$ with

$$|x-y| < \delta \quad \text{and}$$

$$|\sin n \cdot 0 - \sin n \cdot \tfrac{\pi}{2n}| = |0-1| = 1 > \tfrac{1}{2}$$

which is a contradiction.

$\therefore \{\sin nx\}_{n=1}^{\infty}$ is not equicontinuous. #

Lemma 4.3 Let $A = \{z_j\}_{j=1}^{\infty}$ be a countable set and $f_n : A \to \mathbb{R}$, $n = 1, 2, \cdots$, be a sequence of functions defined on A. Suppose that for each $z_j \in A$, $\{f_n(z_j)\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$. Then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $\forall\, z_j \in A$,

$$\{f_{n_k}(z_j)\} \text{ is convergent.}$$

Pf: Since $f_n(z_1)$ is a bounded sequence (in $\mathbb{R}$), $\exists$ a subsequence f_n^1 such that

$$f_n^1(z_1) \text{ is convergent.}$$

(Note that we have used the same index n to denote the subsequence f_{n_k}.
The superscript 1 is to denote that it is convergent when evaluated at z_1.)

For this subsequence f_n^1 (of original f_n),

$f_n^1(z_2)$ is bounded (since $\{f_n^1(z_2)\} \subset \{f_n(z_2)\}$).

Hence $\exists$ a subsequence

$\{f_n^2\}$ of $\{f_n^1\}$ such that

$\{f_n^2(z_2)\}$ is convergent.

Note that since $\{f_n^1\}$ is a subseq. of $\{f_n\}$,

$\{f_n^2\}$ is also a subsequence of $\{f_n\}$.

Also, $\{f_n^2(z_1)\}$ is a subseq. of the

convergent subsequence $\{f_n^1(z_1)\}$,

$\{f_n^2(z_1)\}$ is also convergent.

Therefore, we've found a subseq. $\{f_n^2\}$ of $\{f_n\}$

such that $\{f_n^2(z_1)\}$ and

$\{f_n^2(z_2)\}$ are convergent.

And $\{f_n^2\}$ is a subseq. of $\{f_n^1\}$.

Repeating the process, one can obtain sequences

$\{f_n^j\}$ (with $f_n^0 = f_n$) such that

(i) $\{f_n^{j+1}\}$ is a subsequence of $\{f_n^j\}$, $\forall j = 0, 1, 2, \cdots$

(ii) $\{f_n^j(z_1)\}, \{f_n^j(z_2)\}, \cdots, \{f_n^j(z_j)\}$

are convergent $(j \geq 1)$

$$
\begin{array}{cccccc|c}
 & & & & & & \text{convergent at} \\
f_1^1 & f_2^1 & f_3^1 & \cdots & f_n^1 & \cdots & z_1 \\
f_1^2 & f_2^2 & f_3^2 & \cdots & f_n^2 & \cdots & z_1, z_2 \\
f_1^3 & f_2^3 & f_3^3 & \cdots & f_n^3 & \cdots & z_1, z_2, z_3 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
f_1^n & f_2^n & f_3^n & \cdots & f_n^n & \cdots & z_1, z_2, \cdots, z_n \\
\vdots & \vdots & \vdots & & \vdots & &
\end{array}
$$

Define $g_n = f_n^n$, $\forall n \geq 1$. (the diagonal sequence)

then $\{g_n\}$ is a subsequence of $\{f_n\}$ and

for any fixed $j = 1, 2, \cdots$,

$$g_n(z_j) = f_n^n(z_j)$$

As $n \to \infty$, $n \geq j$ for sufficiently large n.

Hence $\{f_n^n(z_j)\}$ is a subsequence of the convergent sequence $\{f_n^j(z_j)\}$ for all sufficiently large n. Therefore $\{g_n(z_j)\}$ is convergent. This completes the proof of the Lemma. ※

(This method of finding g_n is called Cantor's diagonal trick.)