MATH3060 HW4 Due date: Oct 22, 2020 (at 12:00 noon)

(1) Let (X, d) be a metric space.

(a) If A is a closed subset of X and $x_0 \in X \setminus A$. Show that there is a continuous function f on X such that $f(x_0) = 1$ and $f(x) = 0$, $\forall x \in A$.

(b) If A, B are disjoint closed subsets of X, show that there exists a continuous function g on X such that

$$g(x) = 1, \ \forall x \in A \quad \text{and} \quad g(x) = 0, \ \forall x \in B.$$

(c) A, B as in part (b), show that there exist disjoint open sets G_1 and G_2 such that $A \subset G_1$ and $B \subset G_2$.

(2) Show that $\Psi : (C[-1,1], d_1) \to \mathbb{R}$ given by $\Psi(f) = f(0)$ is not a continuous mapping between metric spaces. ($\mathbb{R}$ always equipped with the standard metric $d(x,y) = |x-y|$, $\forall x, y \in \mathbb{R}$)

(3) Recall that $l_2 = \{ x = (x_1, x_2, \cdots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty, \ x_i \in \mathbb{R} \}$

has a metric $d_2(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{1/2}$. Show that

the set $H = \{ x = (x_1, x_2 \cdots) : |x_i| \leq \frac{1}{i}, \ \forall i = 1, \cdots, \infty \}$

is a closed subset in (l_2, d_2).

(4) Prove the generalized Hölder inequality:

$\forall f_i \in R[a,b], \ i = 1, \cdots, n,$

$$\int_a^b |f_1 f_2 \cdots f_n| dx \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_n\|_{p_n}$$

where $p_i > 1$, for all $i = 1, \cdots, n$, and satisfy $\sum_{i=1}^{n} \frac{1}{p_i} = 1$.

(5) Show that if $p_2 > p_1 \geq 1$, then there exists a constant $C > 0$ such that

$$\|f\|_{p_1} \leq C \|f\|_{p_2} \text{ for all } f \in R[a,b].$$

(End)