Ch3 The Contraction Mapping Principle

§3.1 Complete Metric Space

Def: Let (X, d) be a metric space.

(1) A sequence $\{x_n\}$ in (X, d) is a Cauchy sequence if $\forall \varepsilon > 0$, $\exists n_0$ s.t. $d(x_n, x_m) < \varepsilon$, $\forall n, m \geq n_0$.

(2) (X, d) is complete if every Cauchy sequence in (X, d) converges.

(3) A subset E is complete if the induce metric subspace (E, d) is complete. (i.e. $d = d|_{E\times E}$)
i.e. every Cauchy sequence in E converges with limit in E.

Note: Convergent sequence is a Cauchy sequence (Ex!)

Prop 3.1 Let (X, d) be a metric space.

(a) Every complete set in X is closed.

(b) If X is complete, then every closed set in X is complete.

Pf: (a) Let $E \subset X$, & E complete.

Suppose $\{x_n\} \subset E$ with $x_n \to x$ in X.

By note, $\{x_n\}$ is a Cauchy seq. in E

Then completeness of $E \Rightarrow$ $x_n \to z \in E$.

Uniqueness of limit $\Rightarrow$ $x = z \in E$

$\therefore$ E is closed.

(b) Let (X, d) be complete & E is closed in X.

Then every Cauchy seq. $\{x_n\}$ in E is a Cauchy seq in X. Completeness of $X \Rightarrow \exists x \in X$, s.t. $x_n \to x$. Since E is closed, $x \in E$.

$\therefore$ E is complete.

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eg 3.1:
- ($\mathbb{R}$, standard) is complete
- $[a,b]$, $(-\infty, b]$, $[a, \infty)$ complete
- $[a,b)$ (b finite) not complete ($\because x_n = b - \frac{1}{n} \to b \notin [a,b)$)
- $\mathbb{Q}$ is not complete.

eg 3.2 $(X = C[a,b], d_\infty)$ is complete:

Cauchy seq $\{f_n\}$ in d_∞-metric

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists n_0$ s.t.

$$\max_{[a,b]} |f_n(x) - f_m(x)| < \varepsilon, \quad \forall n, m \geq n_0$$

$\therefore$ $f_n(x) \to f(x)$ uniformly for some $f \in C[a,b]$ ※

eg Let $P = \{ f \in C[a,b] : f(x) = p(x) \text{ on } [a,b] \text{ for some polynomial } p(x) \}$.

Then P is not complete (in d_∞-metric):

$$h_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \in P$$

but $h_n(x) \to e^x$ uniformly (in d_∞-metric)

& $e^x \notin P$.

3.5 Appendix: Completion of a Metric Space

Def: A metric space (X, d) is said to be <u>isometrically embedded</u> in metric space (Y, ρ) if $\exists$ a mapping $\Phi: X \to Y$ s.t.

$$d(x,y) = \rho(\Phi(x), \Phi(y)).$$

Notes: (i) Φ is called an <u>isometric embedding</u> from (X,d) into (Y,ρ). And sometime called a <u>metric preserving map</u>.

(ii) Φ must be one-to-one and continuous.

Def: Let (X,d) and (Y,ρ) be metric spaces. We call (Y,ρ) a <u>completion</u> of (X,d) if (1) (Y,ρ) is complete.

(2) $\exists$ isometric embedding $\Phi: (X,d) \to (Y,\rho)$ such that the closure $\overline{\Phi(X)} = Y$.

eg: $(Y,\rho) = (\mathbb{R}, \text{standard})$, $X = \mathbb{Q} \subset \mathbb{R}$

$(X, d) = (\mathbb{Q}, \text{induced metric})$

Then • $(\mathbb{R}, \text{standard})$ is complete;

• $\Phi : (\mathbb{Q}, \text{induced metric}) \to (\mathbb{R}, \text{standard})$

$$q \mapsto q$$

• & $\overline{\mathbb{Q}} = \mathbb{R}$ ($\mathbb{Q}$ is dense in $\mathbb{R}$)

$\overline{\Phi(\mathbb{Q})}$

Thm 3.2 Every metric space has a completion.

Pf (Sketch of Proof)

Let (X, d) be metric space.

Let $\mathcal{C} = \{ \{x_n\} \subset X : \{x_n\} \text{ Cauchy sequence} \}$

Define equivalent relation $\sim$ on $\mathcal{C}$ by

$$\{x_n\} \sim \{y_n\} \iff d(x_n, y_n) \to 0 \text{ as } n \to \infty.$$

Let $\tilde{\mathcal{C}} = \mathcal{C}/\sim$ the quotient space.

Define $\tilde{d} = \tilde{\mathcal{C}} \times \tilde{\mathcal{C}} \to \mathbb{R}$ by the following:

For $\tilde{x} =$ equi. class $[\{x_n\}]$

$\tilde{y} =$ equi. class $[\{y_n\}]$,

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} d(x_n, y_n)$$

Then $\tilde{d}$ is well-defined and is a metric on $\tilde{\mathcal{C}}$.

One then proves $(\tilde{\mathcal{C}}, \tilde{d})$ is complete.

$$\begin{array}{ccc} \Phi : (X, d) & \longrightarrow & (\tilde{\mathcal{C}}, \tilde{d}) \\ \cup\!\!\!\mid & & \cup\!\!\!\mid \\ x & \longmapsto & [\{x, x, x, \cdots\}] \end{array}$$ defined by

is an isometric embedding.

And one can show that

$$\overline{\Phi(X)}^{\text{closure in } (\tilde{\mathcal{C}}, \tilde{d})} = \tilde{\mathcal{C}}.$$

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Def: Two metric spaces (X, d), (X', d') are called <u>isometric</u> if $\exists$ <u>bijective</u> isometric embedding from (X, d) onto (X', d').

<u>Notes</u> : (i) the inverse of the bijective isometric embedding is also an isometric embedding.

(ii) Two metric spaces will be regarded as the <u>same</u> if they are isometric.

<u>Thm</u> : If (Y, ρ) & (Y', ρ') are completions of a metric space (X, d). Then (Y, ρ) and (Y', ρ') are isometric.

i.e. Completion is unique up to isometry.

§3.2 The Contraction Mapping Principle

Def : (1) Let (X, d) be a metric space. A map $T: (X,d) \to (X,d)$ is called a contraction if $\exists$ constant $\gamma \in (0,1)$, such that $d(Tx, Ty) \leq \gamma d(x,y)$, $\forall x, y \in X$.

(2) A point $x \in X$ is called a fixed point of T if $Tx = x$.

(Usually we write Tx instead of $T(x)$.)

Thm 3.3 (Contraction Mapping Principle)

Every contraction in a complete metric space admit a unique fixed point.

(This is also called the Banach Fixed Point Thm)

Pf: <u>Uniqueness</u>: Suppose x & y are fixed pts. of T. Then

$$d(x,y) = d(Tx, Ty) \quad (x, y \text{ are fixed by } T)$$
$$\leq \gamma\, d(x,y) \text{ for some } \gamma \in (0,1).$$
$$(T \text{ contraction})$$

$$\Rightarrow d(x,y) = 0 \Rightarrow x = y.$$

<u>Existence</u>: Let $x_0 \in X$.

Define $\{x_n\}_{n=1}^{\infty}$ by $x_n = Tx_{n-1}$, for $n = 1, 2, \ldots$

Then $x_n = Tx_{n-1} = T(Tx_{n-2}) = T^2 x_{n-2}$

$$= \cdots = T^n x_0.$$

For any $n \geq N$,

$$d(x_n, x_N) = d(T^n x_0, T^N x_0) = d(T^{(n-N)+N} x_0, T^N x_0)$$
$$= d(T(T^{(n-N)+N-1} x_0), T(T^{N-1} x_0))$$
$$\leq \gamma\, d(T^{(n-N)+N-1} x_0, T^{N-1} x_0)$$

(where $\gamma \in (0,1)$ is the constant s.t. $d(Tx, Ty) \leq \gamma d(x,y)$, $\forall x, y \in X$)

$$\leq \cdots$$

$$\leq \gamma^N d(T^{(n-N)} x_0, x_0)$$

$$\leq \gamma^N \Big[d(T^{(n-N)} x_0, T^{(n-N)-1} x_0) + d(T^{(n-N)-1} x_0, T^{(n-N)-2} x_0) + \cdots + d(Tx_0, x_0) \Big]$$

$$\leq \gamma^N \Big[d(Tx_0, x_0) + \gamma d(Tx_0, x_0) + \cdots + \gamma^{(n-N)-2} d(Tx_0, x_0) + \gamma^{(n-N)-1} d(Tx_0, x_0) \Big]$$

$$= \gamma^N \Big[1 + \gamma + \cdots + \gamma^{(n-N)-1} \Big] d(Tx_0\, x_0)$$

$$< \frac{\gamma^N}{1-\gamma} d(Tx_0, x_0)$$

Therefore, $\forall \varepsilon > 0$, if $N > 0$ is chosen s.t.

$$\frac{\gamma^N}{1-\gamma} d(Tx_0, x_0) < \frac{\varepsilon}{2},$$

we have $\forall\, n, m \geq N$,

$$d(x_n, x_m) \leq d(x_n, x_N) + d(x_N, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore$ $\{x_n\}$ is a Cauchy seq. in (X, d).

By completeness of $(\mathbb{X}, d)$, $\exists\, x \in \mathbb{X}$ s.t.

$$x_n \to x.$$

Note that a contraction is always continuous (Ex!)

we have

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = T \lim_{n\to\infty} x_{n-1} = Tx.$$

$\therefore$ x is a fixed point of T. ※

<u>eg 3.3</u> $\quad T : (0,1] \to (0,1]$ (Caution: $(0,1]$ is not complete)

$$x \mapsto \frac{x}{2}.$$

Clearly $\quad |Tx - Ty| = \frac{1}{2}|x-y| \quad r = \frac{1}{2} < 1$

$\therefore$ T is a contraction.

However, if $x \in (0,1]$ is a fixed point of T,

then $\quad Tx = x \iff \frac{x}{2} = x \iff x = 0 \notin (0,1]$.

$\therefore$ T has <u>no</u> fixed point on $(0,1]$.

This example shows that "<u>completeness</u>" is necessary in the Contraction Mapping Principle.

eg 3.4: $S : \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ is complete)

$x \mapsto x - \log(1+e^x)$.

Then $$\frac{dS}{dx} = 1 - \frac{e^x}{1+e^x} = \frac{1}{1+e^x} > 0$$

$$\Rightarrow \quad |Sx - Sy| = \left|\frac{dS}{dx}(c)\right| |x-y| < |x-y|$$

(But there is no constant $\gamma < 1$ such that $|Sx - Sy| \le \gamma |x-y|$ (Ex!))

Since $-\log(1+e^x) \neq 0 \ \forall x \in \mathbb{R}$, $Sx \neq x \ \forall x \in \mathbb{R}$
no fixed point

This example shows that $\gamma < 1$ cannot replaced by $\gamma \le 1$ #

§3.3 The Inverse Function Theorem

Notation: Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at a point p in an open set U of $\mathbb{R}^n$. We write

$$F = \begin{pmatrix} f^1 \\ \vdots \\ f^m \end{pmatrix} \in \mathbb{R}^m,$$

where $\quad f^i = f^i(x^1, \cdots, x^n) : U \to \mathbb{R}, \ \forall i = 1, \cdots, m.$

Then F differentiable at $p_0 = \begin{pmatrix} x_0^1 \\ \vdots \\ x_0^n \end{pmatrix} \in U \subset \mathbb{R}^n$

$\Rightarrow$

$$F(p_0 + z) - F(p_0) = DF(p_0)z + o(z)$$

$$\forall \ z = \begin{pmatrix} z^1 \\ \vdots \\ z^n \end{pmatrix} \text{ sufficiently small,}$$
$$(|z| \text{ small})$$

where

$$DF(p_0) = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n} \end{pmatrix}$$

or $\begin{pmatrix} f_1^1 & \cdots & f_n^1 \\ \vdots & & \vdots \\ f_1^m & \cdots & f_n^m \end{pmatrix}$

i.e. $$\left(DF(p_0)z\right)^i = \sum_{j=1}^{n} \frac{\partial f^i}{\partial x^j}(p_0) z^j \quad \forall i = 1, \cdots, m.$$