§2.5 Elementary Inequalities for Functions

Recall

Young's Inequality

For $a, b > 0$ and $p > 1$,

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{where } q \text{ is given by } \frac{1}{p} + \frac{1}{q} = 1$$

and "equality holds" $\iff a^p = b^q$.

Note: $q = \frac{p}{p-1} > 1$ is called the conjugate of p.

(Recall Pf: Study the minimum of

$$\varphi(a) = \frac{a^p}{p} + \frac{b^q}{q} - ab. \quad (\text{Ex!})$$
)

Note: If $p = 2$, it is the elementary inequality

$$2ab \le a^2 + b^2.$$

Thm 2.10 (Hölder's Inequality)

Let $f, g \in R[a,b]$ (Riemann integrable) and $p > 1$.

Then $$\int_a^b |f(x)g(x)|\,dx \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}},$$

where $q = \frac{p}{p-1}$ the conjugate of p.

"Equality holds"

$\iff$ either (a) f or $g = 0$ almost everywhere,

or (b) $\exists$ constant $\lambda > 0$ s.t.

$$|g(x)|^q = \lambda |f(x)|^p \text{ almost everywhere.}$$

($\iff$ $\exists$ constants $\lambda_1, \lambda_2 \geq 0$, not both zero, such that

$\lambda_1 |f(x)|^p = \lambda_2 |g(x)|^q$ a.e.)

Note: If we denote $\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$. Then

the Hölder Inequality can be written as

$$\int_a^b |f(x)g(x)|\,dx \leq \|f\|_p \|g\|_q .$$

Pf: If $\|f\|_p = 0$, or $\|g\|_q = 0$. Then $f=0$ or $g=0$ almost everywhere. Hence

$$0 = \int_a^b |f(x)g(x)|dx$$

and the inequality holds trivially.

Assume now that $\|f\|_p > 0$ and $\|g\|_q > 0$.

By Young's Inequality, we have for any $\varepsilon > 0$,

$$|f(x)g(x)| = \left|\varepsilon f(x) \cdot \frac{g(x)}{\varepsilon}\right|$$

$$\leq \frac{\varepsilon^p |f(x)|^p}{p} + \frac{|g(x)|^q}{q\varepsilon^q}, \quad \forall x \in [a,b]$$

$$\Rightarrow \int_a^b |f(x)g(x)|dx \leq \frac{\varepsilon^p}{p}\int_a^b |f(x)|^p dx + \frac{1}{q\varepsilon^q}\int_a^b |g(x)|^q dx$$

$$= \frac{\varepsilon^p}{p}\|f\|_p^p + \frac{1}{q\varepsilon^q}\|g\|_q^q$$

Choose $\varepsilon > 0$ s.t.

$$\varepsilon^p \|f\|_p^p = \frac{1}{\varepsilon^q} \|g\|_q^q$$

i.e. $$\varepsilon^{p+q} = \frac{\|g\|_q^q}{\|f\|_p^p} \left(= \frac{\int_a^b |g(x)|^q dx}{\int_a^b |f(x)|^p dx} \right)$$

i.e. $$\varepsilon = \frac{\|g\|_q^{\frac{q}{p+q}}}{\|f\|_p^{\frac{p}{p+q}}} > 0 .$$

Then

$$\int_a^b |f(x)g(x)| dx \leq \frac{\varepsilon^p}{p} \|f\|_p^p + \frac{1}{q\varepsilon^q} \|g\|_q^q$$

$$= \left(\frac{1}{p} + \frac{1}{q}\right) \varepsilon^p \|f\|_p^p$$

$$= \frac{\|g\|_q^{\frac{pq}{p+q}}}{\|f\|_p^{\frac{p^2}{p+q}}} \cdot \|f\|_p^p \qquad \left(\begin{array}{c}\text{using } \frac{1}{p}+\frac{1}{q}=1 \\ \Updownarrow \\ \frac{p+q}{pq}=1\end{array}\right)$$

$$= \|g\|_q \, \|f\|_p^{p\left(1-\frac{p}{p+q}\right)}$$

$$= \|g\|_q \, \|f\|_p^{p \cdot \frac{q}{p+q}}$$

$$= \|g\|_q \, \|f\|_p .$$

From the proof, "Equality holds"

$$\Leftrightarrow \quad |f(x)g(x)| = \frac{\varepsilon^p |f(x)|^p}{p} + \frac{|g(x)|^q}{q\,\varepsilon^q}$$

almost everywhere

(with ε given above)

$$\Leftrightarrow \quad \varepsilon^p |f(x)|^p = \frac{|g(x)|^q}{\varepsilon^q} \quad \text{almost everywhere}$$

$$\Leftrightarrow \quad |g(x)|^q = \varepsilon^{p+q} |f(x)|^p \quad \text{almost everywhere.}$$

$$\Rightarrow \exists \, \lambda = \varepsilon^{p+q} > 0 \text{ s.t. } |g(x)|^q = \lambda |f(x)|^p \text{ a.e.}$$

Conversely, if $|g(x)|^q = \lambda |f(x)|^p$ for some $\lambda > 0$,

we clearly have the "Equality". #

Note: Limiting cases (Note: Riemann integrable $\Rightarrow$ bounded)

(i) $p \to 1$ ($\Rightarrow q \to +\infty$)

$$\int_a^b |f(x)g(x)|\,dx \le \|f\|_1 \|g\|_\infty$$

(ii) $p \to +\infty$ ($\Rightarrow q \to 1$)

$$\int_a^b |f(x)g(x)|\,dx \le \|f\|_\infty \|g\|_1$$

$$\left(= \|f\|_\infty \int_a^b |g(x)|\,dx \right)$$

Thm 2.11 (Minkowski's Inequality)

$\forall f, g \in R[a,b]$, and $p > 1$,

$$\|f+g\|_p \le \|f\|_p + \|g\|_p .$$

"Equality holds"

$\Leftrightarrow$ either (a) f or $g = 0$ a.e.

or (b) $\|f\|_p > 0$, $\|g\|_p > 0$ and $\exists$ constant $\lambda > 0$ s.t. $g(x) = \lambda f(x)$ a.e.

($\Leftrightarrow$ $\exists$ constants $\lambda_1, \lambda_2 \ge 0$, not both zero, s.t. $\lambda_1 f(x) = \lambda_2 g(x)$ a.e.)

Pf: Clearly

$$|f+g|^p = |f+g|^{p-1}|f+g|$$
$$\leq |f+g|^{p-1}|f| + |f+g|^{p-1}|g|$$

$$\Rightarrow \int_a^b |f+g|^p \leq \int_a^b |f+g|^{p-1}|f| + \int_a^b |f+g|^{p-1}|g|$$

Hölder's Inequality $\Rightarrow$

$$\int_a^b |f+g|^p \leq \left(\int_a^b |f|^p\right)^{\frac{1}{p}} \left[\int_a^b \left(|f+g|^{p-1}\right)^q\right]^{\frac{1}{q}}$$
$$+ \left(\int_a^b |g|^p\right)^{\frac{1}{p}} \left[\int_a^b \left(|f+g|^{p-1}\right)^q\right]^{\frac{1}{q}}$$

Since $q = \frac{p}{p-1}$, we have $q(p-1) = p$.

$$\therefore \int_a^b |f+g|^p \leq \left(\|f\|_p + \|g\|_p\right)\left(\int_a^b |f+g|^p\right)^{\frac{1}{q}}$$

$$\Rightarrow \left(\int_a^b |f+g|^p\right)^{1-\frac{1}{q}} \leq \|f\|_p + \|g\|_p$$

$$\Rightarrow \quad \left(\int_a^b |f+g|^p\right)^{\frac{1}{p}} \le \|f\|_p + \|g\|_p .$$

$\left(\frac{1}{p}+\frac{1}{q}=1\right)$

i.e. $$\|f+g\|_p \le \|f\|_p + \|g\|_p .$$

"Equality holds" $\Longleftrightarrow$

(i) $|f+g|^{p-1}|f+g| = |f+g|^{p-1}(|f|+|g|)$ a.e.

(ii) "Equality holds" in Hölder's inequality for

$$\int_a^b |f+g|^{p-1}|f| \le \left(\int_a^b |f|^p\right)^{\frac{1}{p}} \left[\int_a^b \left(|f+g|^{p-1}\right)^q\right]^{\frac{1}{q}}$$

$$\int_a^b |f+g|^{p-1}|g| \le \left(\int_a^b |g|^p\right)^{\frac{1}{p}} \left[\int_a^b \left(|f+g|^{p-1}\right)^q\right]^{\frac{1}{q}}$$

which implies $\begin{cases} |f+g|^{(p-1)q} = \lambda_1 |f|^p & \text{and} \\ |f+g|^{(p-1)q} = \lambda_2 |g|^p & \text{(a.e.)} \end{cases}$

for some constants $\lambda_1 > 0$, $\lambda_2 > 0$.

Here we assumed "$|f+g|=0$ a.e." is not true.

Otherwise, "$|f+g|=0$ a.e." $\Rightarrow \|f+g\|_p = 0$.

"equality holds" $\Rightarrow$ $\|f\|_p + \|g\|_p \Rightarrow$ f & $g = 0$ a.e.
which is the trivial case.

Hence $|f|^p = \frac{\lambda_2}{\lambda_1} |g|^p$ a.e.
Which implies $f(x) = \pm \lambda\, g(x)$ a.e.

where $\lambda = \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{p}} > 0$

"$|f+g| = 0$ a.e." is not true.

(either $\{x : |f+g| > 0\}$ has "measure" > 0
or $\{x : |f+g| > 0\}$ is "not measurable"
Using Lebseque theory (omitted), $\{x : |f+g| > 0\}$
is measurable.)

Hence $\Omega_1 = \{x : |f+g| > 0\}$ has "measure" > 0.

and for $x \in \Omega_1$, we can use (i) to obtain that

$$|f(x) + g(x)| = |f(x)| + |g(x)| \text{ a.e. in } \Omega_1$$

and hence

$$f(x) = \lambda g(x) \text{ a.e in } \Omega_1$$

For $\forall\, x \notin \Omega_1$, i.e. $|f(x)+g(x)|=0$.

Then $\begin{cases} |f+g|^p = \lambda_1 |f|^p \ \& \\ |f+g|^p = \lambda_2 |g|^p \end{cases}$ a.e.

$\Rightarrow$ $f=0$ and $g=0$ a.e in $[a,b]\setminus\Omega_1$

Hence trivially $f=\lambda g$ a.e. in $[a,b]\setminus\Omega_1$.

Together, we have

$$f(x) = \lambda\, g(x) \quad \text{a.e. in } [a,b]$$

#

<u>Alternate proof</u> (not using Hölder's inequality)

<u>Lemma</u>: $\varphi(t) = t^p$, $p>1$, is strictly convex on $[0,\infty)$.

(in fact, $\varphi''(t)>0$, $\forall\, t\in(0,\infty)$)

Hence $\forall\, a,b>0$ and $\lambda\in[0,1]$,

$$[(1-\lambda)a+\lambda b]^p \leq (1-\lambda)a^p + \lambda b^p$$

"Equality holds" $\Leftarrow$ $a=b$

Pf :(Sketch)

Consider $\psi(\lambda) = [(1-\lambda)a + \lambda b]^p - (1-\lambda)a^p - \lambda b^p$, for $\lambda \in [0,1]$

Use $\varphi''(t) > 0$ to show that

$$\psi''(\lambda) > 0 \quad \text{for } \lambda \in (0,1]$$

Hence maximum of ψ attained at the boundary $\lambda = 0$ & $\lambda = 1$.

Since $\psi(0) = 0 = \psi(1)$,

$$\psi(\lambda) \leq 0 \ , \ \forall \lambda \in [0,1] .$$

This also shows the "equality" case. $\#$

Pf of Minkowski's Inequality

If $\|f\|_p = 0$, then $f(x) = 0$ a.e. $x \in [a,b]$, the inequality in fact an equality.

Similarly if $\|g\|_p = 0$.

Assume $\|f\|_p > 0$ and $\|g\|_p > 0$.

Then $\int_a^b |f+g|^p(x)\,dx$

$$\leq \int_a^b (|f|+|g|)^p(x)\,dx \quad (\text{since } p>1)$$

$$= (\|f\|_p+\|g\|_p)^p \int_a^b \left[\frac{\|f\|_p}{\|f\|_p+\|g\|_p} \cdot \frac{|f(x)|}{\|f\|_p} + \frac{\|g\|_p}{\|f\|_p+\|g\|_p} \cdot \frac{|g(x)|}{\|g\|_p} \right]^p dx$$

using the fact that $\varphi(t)=t^p$ $t\in[0,+\infty)$ is a strictly convex function (for $p>1$)

and $\dfrac{\|f\|_p}{\|f\|_p+\|g\|_p} + \dfrac{\|g\|_p}{\|f\|_p+\|g\|_p} = 1$, we have

$$\int_a^b |f+g|^p(x)\,dx$$

$$\leq (\|f\|_p+\|g\|_p)^p \int_a^b \left[\frac{\|f\|_p}{\|f\|_p+\|g\|_p} \left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{\|g\|_p}{\|f\|_p+\|g\|_p} \left(\frac{|g(x)|}{\|g\|_p}\right)^p \right] dx$$

$$= (\|f\|_p+\|g\|_p)^p \left[\frac{\|f\|_p}{\|f\|_p+\|g\|_p} \cdot \frac{\int_a^b |f(x)|^p dx}{\|f\|_p^p} + \frac{\|g\|_p}{\|f\|_p+\|g\|_p} \frac{\int_a^b |g(x)|^p dx}{\|g\|_p^p} \right]$$

$$= (\|f\|_p+\|g\|_p)^p \left[\frac{\|f\|_p}{\|f\|_p+\|g\|_p} + \frac{\|g\|_p}{\|f\|_p+\|g\|_p} \right]$$

$$= (\|f\|_p+\|g\|_p)^p.$$

$\therefore \quad \|f+g\|_p \le \|f\|_p + \|g\|_p .$

Equality holds: (if $\|f\|_p > 0$, $\|g\|_p > 0$)

$\Leftrightarrow$ (1) $|f+g| \le |f| + |g|$ a.e. (No $|f+g|^{p-1}$ facta)

(2) $\dfrac{|f(x)|}{\|f\|_p} = \dfrac{|g(x)|}{\|g\|_p}$ a.e. (By strict convexity of t^p, $p>1$)

It is clear now $f(x) = \lambda g(x)$ a.e. $x \in [a,b]$

for some positive constant $\lambda = \dfrac{\|f\|_p}{\|g\|_p} > 0$. #

Remark: This shows that $\|f\|_p$ for $p>1$ is

a norm on $R[a,b]/\sim$ ($\leftarrow$ relation mod a.e.)

(other conditions are trivial)