eg 2.14: Let $X \neq \emptyset$ and d = discrete metric on X.

Then $\forall$ subset $E \subset X$,

$$B_{\frac{1}{2}}(x) = \{x\} \subset E, \quad \forall x \in E.$$

$\therefore$ E is open.

Therefore, any subset E of $(X, \text{discrete})$ is open,

& hence any subset E of $(X, \text{discrete})$ is closed.

Together, any subset E of $(X, \text{discrete})$ is both open and closed.

In particular, any $\{x\} \subset (X, \text{discrete})$ is both open and closed.

Prop 2.6 Let (X, d) be a metric space. A sequence $\{x_n\}$ converges to x if and only if

$\forall$ open set G containing x, $\exists n_0$ such that

$$x_n \in G, \quad \forall n \geq n_0.$$

Pf: ($\Rightarrow$) Let G open & $x \in G$

$\Rightarrow \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset G$

As $x_n \to x$, for this $\varepsilon > 0$, $\exists n_0$ s.t.

$d(x_n, x) < \varepsilon$, $\forall n \geq n_0$

$\Rightarrow x_n \in B_\varepsilon(x) \subset G$, $\forall n \geq n_0$

($\Leftarrow$) $\forall \varepsilon > 0$, $B_\varepsilon(x)$ is an open set containing x.

Therefore $\exists n_0$ s.t. $x_n \in B_\varepsilon(x)$, $\forall n \geq n_0$

$\Rightarrow d(x_n, x) < \varepsilon$, $\forall n \geq n_0$

$\#$

Prop 2.7 Let (X, d) be a metric space. Then a set $A \subset X$ is closed if and only if whenever $\{x_n\} \subset A$ and $x_n \to x$ as $n \to \infty$ implies that $x \in A$.

Pf: ($\Rightarrow$) Suppose not. Then $x \notin A$

i.e. $x \in X \setminus A$ which is open (as A closed)

$\Rightarrow \exists \varepsilon > 0$, $B_\varepsilon(x) \subset X \setminus A$.

On the other hand $x_n \to x$, $\exists n_0$ s.t. $d(x_n, x) < \varepsilon$ $\forall n \geq n_0$

$\Rightarrow x_n \in B_\varepsilon(x) \subset X \setminus A$

$\Rightarrow x_n \notin A$ contradiction $\#$

($\Leftarrow$) Suppose not. Then A is not closed.

$\Leftrightarrow$ $X \setminus A$ is not open

$\exists\, x \in X \setminus A$ s.t. $B_\varepsilon(x) \not\subset X \setminus A$, $\forall \varepsilon > 0$.

In particular, $B_{\frac{1}{n}}(x) \cap A \neq \emptyset$, $\forall n = 1, 2, \cdots$

Pick $x_n \in B_{\frac{1}{n}}(x) \cap A$ for each n

Then $\{x_n\} \subset A$ & $d(x_n, x) < \frac{1}{n}$, $\forall n$

$\Rightarrow x_n \to x$ as $n \to \infty$.

Contradicting the assumption (as $x \in X \setminus A$.) ※

Prop 2.8 Let $f : (X, d) \to (Y, \rho)$ be a mapping between metric spaces,

(a) f is continuous at x

$\Leftrightarrow$ $\forall$ open set G (in Y) containing $f(x)$,

$f^{-1}(G)$ contains $B_\varepsilon(x)$ for some $\varepsilon > 0$.

(b) f is continuous in X

$\Leftrightarrow$ $\forall$ open set G in Y, $f^{-1}(G)$ is open in X

Pf: (a) ($\Rightarrow$) Suppose not,

then $\exists$ open set G in Y containing $f(x)$

s.t. $f^{-1}(G)$ doesn't containing $B_\varepsilon(x)$, $\forall \varepsilon>0$.

i.e. $B_\varepsilon(x) \cap [X \setminus f^{-1}(G)] \neq \phi$, $\forall \varepsilon > 0$.

In particular $B_{\frac{1}{n}}(x) \cap [X \setminus f^{-1}(G)] \neq \phi$, $\forall n$.

Pick $x_n \in B_{\frac{1}{n}}(x) \cap [X \setminus f^{-1}(G)]$, $\forall n$.

Then

$$\begin{cases} x_n \in B_{\frac{1}{n}}(x) \Rightarrow x_n \to x \text{ as } n \to \infty \\ x_n \in X \setminus f^{-1}(G) \Rightarrow f(x_n) \notin G, \forall n \end{cases}$$

By Prop 2.5, $f(x_n) \not\to f(x)$. Contradicting the assumption that f is cts. at x.

($\Leftarrow$) $\forall \varepsilon > 0$, $B_\varepsilon(f(x)) \subset Y$ is an open set containing $f(x)$. By assumption,

$$f^{-1}(B_\varepsilon(f(x))) \supset B_\delta(x) \text{ for some } \delta > 0$$

i.e $f(y) \in B_\varepsilon(f(x))$, $\forall y \in B_\delta(x)$

$$\Rightarrow \quad d(f(y), f(x)) < \varepsilon, \quad \forall\, d(y,x) < \delta.$$

$\therefore$ f is cts. at x.

(b) follows from (a). (Ex!) #

<u>Note</u>: We also have:

f is cts in X

$\Longleftrightarrow$ $\forall$ closed set $F \subset Y$, $f^{-1}(F)$ is closed in X.

(Pf: Ex!)

<u>Eg</u>: (i) Let $A \subset X$ & $A \neq \emptyset$.

Since $\rho_A(x) = d(x, A)$ is cts,

$$G_r = \{x \in X : d(x, A) < r\} = \rho_A^{-1}(\underbrace{B_r(0)}_{\text{in } \mathbb{R}})$$

is open in X.

(ii) <u>Claim</u>: If A is closed $(\neq \emptyset)$, then $A = \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$.

Hence any closed set is a <u>countable intersection</u> of open sets.

Pf : It is clear that $A \subset \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$ as $A \subset G_{\frac{1}{n}}$, $\forall n$.

Let $x \in \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$ then $x \in G_{\frac{1}{n}}$, $\forall n$

$\Rightarrow$ $d(x, A) < \frac{1}{n}$, $\forall n$

$\Rightarrow$ $\exists\, x_n \in A$ s.t. $d(x, x_n) < \frac{1}{n}$, $\forall n$

Hence $\{x_n\} \subset A$ is a seq in A s.t. $x_n \to x$.

Since A is closed, we have $x \in A$. (Prop 2.7)

$\therefore$ $A = \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$ ※

§2.4 Points in Metric Spaces

Def: Let E be a set in a metric space (X, d)

(1) A point $x \in X$ (not nec. in E) is called a boundary point of E if $\forall$ open set $G \subset X$ containing x, $G \cap E \neq \phi$ & $G \setminus E \neq \phi$
$(G \cap (X \setminus E) \neq \phi)$

(2) The set of boundary points of E will be denoted by ∂E and is called the boundary of E.

(3) The closure of E, denoted by $\overline{E}$, is defined to be $\overline{E} = E \cup \partial E$.

Note:

(i) In (1), it suffices to check G of the form $B_\varepsilon(x)$ for all small $\varepsilon > 0$, or even $B_{\frac{1}{n}}(x)$, $\forall n \geq 1$ (See the proof of Prop 2.9 (a)).

(ii) $\partial E = \partial(X \setminus E)$, $\forall E \subset X$.

eg: For $B_r(x) = \{y \in X : d(y,x) < r\}$ in $(\mathbb{R}^n$, standard$)$

$$\partial B_r(x) = S_r(x) = \{y \in X : d(y,x) = r\} \quad \&$$

$$\overline{B}_r(x) = B_r(x) \cup \partial B_r(x)$$
$$= \{y \in X : d(y,x) \le r\}$$

Further Notes

(i) $\partial \emptyset = \emptyset$ (Ex!)

(ii) $\forall E \subset X$, ∂E is a closed set.

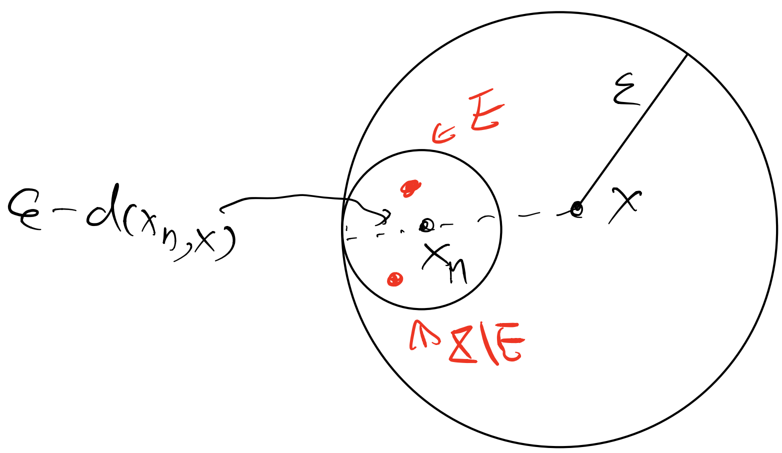
(iii) If E is closed, then $\overline{E} = E$.

Pf of (ii): Consider a seq $\{x_n\} \subset \partial E$ converging to some $x \in X$.

Then $\forall \varepsilon > 0$, $x_n \in B_\varepsilon(x)$ for $n \ge n_0$ (for some n_0)

$\Rightarrow B_{\varepsilon - d(x_n,x)}(x_n) \subset B_\varepsilon(x)$.

As $x_n \in \partial E$,



$$\begin{cases} B_{\varepsilon - d(x_n,x)}(x_n) \cap E \neq \emptyset \\ B_{\varepsilon - d(x_n,x)}(x_n) \setminus E \neq \emptyset \end{cases}$$

$$\Rightarrow \begin{cases} B_\varepsilon(x) \cap E \neq \phi \\ B_\varepsilon(x) \setminus E \neq \phi \end{cases}$$

(since $\varepsilon > 0$ arbitrary) $\Rightarrow$ $x \in \partial E$. Therefore ∂E is closed #

<u>Pf of (iii)</u>: Only need to show that $\partial E \subset E$ if E is closed. Let $x \in \partial E$, then by definition

$$B_{\frac{1}{n}}(x) \cap E \neq \phi \quad (\& \ B_{\frac{1}{n}}(x) \cap (X \setminus E) \neq \phi)$$

$$\Rightarrow \quad \exists \ x_n \in B_{\frac{1}{n}}(x) \cap E.$$

$$\Rightarrow \quad d(x_n, x) < \frac{1}{n}, \quad \forall n$$

$$\therefore \quad x_n \to x$$

Since E is closed, Prop 2.7 $\Rightarrow$ $x \in E$.

Since $x \in \partial E$ is arbitrary, $\partial E \subset E$. #

Prop 2.9 Let $E \subset (X, d)$. Then

(a) $x \in \overline{E} \iff B_r(x) \cap E \neq \emptyset, \ \forall r > 0$.

(b) $A \subset B \Rightarrow \overline{A} \subset \overline{B} \quad \forall A, B \subset (X, d)$

(c) $\overline{E}$ is closed

(d) $\overline{E} = \cap \{C : C = \text{closed set}, \ C \supset E\}$.

(i.e. $\overline{E}$ is the smallest closed set containing E)

Pf (a)($\Rightarrow$) $x \in \overline{E} \Rightarrow x \in E$ or $x \in \partial E$.

If $x \in E$, then $x \in B_r(x) \cap E, \ \forall r > 0$

$\Rightarrow B_r(x) \cap E \neq \phi, \ \forall r > 0$.

If $x \in \partial E$, then by definition of boundary point,

$\forall$ open set G containing x, $G \cap E \neq \phi$

$(\& \ G \setminus E \neq \phi)$

Since $B_r(x)$ is open and $x \in B_r(x), \forall r > 0$,

we have $B_r(x) \cap E \neq \emptyset, \ \forall r > 0$.

($\Leftarrow$) If $x \in E$, we are done. ($x \in E \subset \overline{E}$)

If $x \notin E$, then for any open set G containing x,

$x \in G \setminus E$. Hence $G \setminus E \neq \phi$.

To show that $G \cap E \neq \phi$, we choose $r_0 > 0$

s.t. $B_{r_0}(x) \subset G$ (it is possible since G is open).

Then by assumption, $B_{r_0}(x) \cap E \neq \phi$

and hence $G \cap E \, (\supset B_{r_0}(x) \cap E) \neq \phi$. #

(b) Let $x \in \overline{A}$. By part (a),

$$B_r(x) \cap A \neq \phi, \ \forall r > 0$$

Since $A \subset B$, $B_r(x) \cap B \neq \phi, \ \forall r > 0$

Part (a) again, $x \in \overline{B}$.

$\therefore \ \overline{A} \subset \overline{B}$. #

(c) Consider a seq $\{x_n\} \in \overline{E}$ such that $x_n \to x$

for some $x \in X$. We need to show that $x \in \overline{E}$ (Prop 2.7).

Suppose not, then $x \notin \overline{E}$.

Part (a) $\Rightarrow \exists\ \varepsilon_0 > 0$ such that

$$B_{\varepsilon_0}(x) \cap E = \phi$$

For this $\varepsilon_0 > 0$, $\exists\ n_0 > 0$ such that $x_n \in B_{\varepsilon_0}(x)\ \forall\ n \geq n_0$.

Then $B_{\varepsilon_0}(x) \cap E = \phi \Rightarrow x_n \in \partial E \setminus E$ for $n \geq n_0$.

In particular $\{x_n\}_{n=n_0}^{\infty}$ is a seq. in ∂E and $x_n \to x$. By Note (ii) above and prop 2.7, $x \in \partial E \subset \overline{E}$ which is a contradiction.

#

(d) By (c), $\overline{E}$ is closed & $\overline{E} \supset E$

$\therefore\ \overline{E} \in \{C : C = \text{closed set}, C \supset E\}$

$\Rightarrow\ \overline{E} \supset \cap\{C : C = \text{closed set}, C \supset E\}$

Conversely, let C be a closed set & $C \supset E$.

Then by (b) and (iii) of Further Notes above,

$$\overline{E} \subset \overline{C} = C$$

$$\Rightarrow \quad \overline{E} \subset \cap\{C : C = \text{closed set}, C \supset E\}$$

#

Def: Let E be a subset of a metric space (X, d).

(1) A point x is called an <u>interior point</u> of E if $\exists$ an open set G s.t. $x \in G$ & $G \subset E$.

(2) The set of all interior points of E is called the <u>interior of E</u>, denoted by E°.

<u>Notes</u>: (i) E° is open

(ii) $E^{\circ} = E \setminus \partial E$

(iii) $E^{\circ} = X \setminus \overline{(X \setminus E)}$

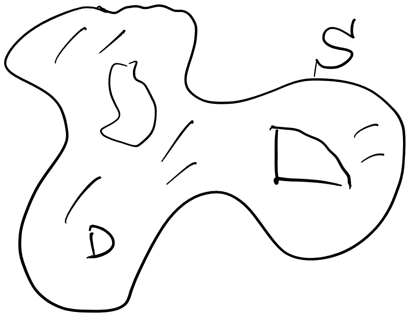
(iv) $E^{\circ} = \bigcup \{ G : G = \text{open} \ \& \ G \subset E \}$

(Pf: Ex!)

<u>eg 2.18</u> $E = \mathbb{Q} \cap [0,1]$ in $(X = [0,1],\ d(x,y) = |x-y|)$

Then $E^{\circ} = \emptyset$ & $\overline{E} = [0,1]$, $\partial E = ?$.

<u>eg 2.19</u> Let D be a domain in $\mathbb{R}^2$ bounded by several cts. curves S.



Then $\partial D = S$

$\overline{D} = D \cup S = D \cup \partial D$

& $(\overline{D})^{\circ} = D$.

eg 2.20 : (i) $\overline{E\cup F} = \overline{E} \cup \overline{F}$ for $E, F \subset (X, d)$
(Ex!)

(ii) However $(E\cup F)^\circ \neq E^\circ \cup F^\circ$ in general.

Counterexample: $E = \mathbb{Q}$, $F = \mathbb{R}\setminus\mathbb{Q}$, $(X, d) = (\mathbb{R}, \text{standard})$

Then $E\cup F = \mathbb{R}$

$\Rightarrow (E\cup F)^\circ = \mathbb{R}$

However $E^\circ = F^\circ = \emptyset$

$\Rightarrow E^\circ \cup F^\circ = \emptyset \neq \mathbb{R} = (E\cup F)^\circ$.

(iii) We only have $E^\circ \cup F^\circ \subset (E\cup F)^\circ$ (Pf: Ex!)

eg 2.21 : $(X = C[0,1],\ d_\infty(f,g) = \|f-g\|_\infty)$

Let $S = \{f \in X : 1 < f(x) \le 5,\ \forall x \in [0,1]\}$

(1) Claim: $\overline{S} = \{f \in X : 1 \le f(x) \le 5,\ \forall x \in [0,1]\}$.

Pf: Let $C = \{f \in X : 1 \le f(x) \le 5,\ \forall x \in [0,1]\}$.

Then $C = \{1 \le f(x)\} \cap \{f(x) \le 5\}$

closed in (X, d_∞)

$\therefore C$ is closed.

$\therefore \overline{S} \subset C$

Conversely, $\forall f \in C$

$$f_n(x) = \max\{f(x), 1+\tfrac{1}{n}\} \in X = C[0,1].$$

$\forall n$.

Then

$$1 < 1+\tfrac{1}{n} \leq f_n(x) \leq 5, \quad \forall n$$

$\Rightarrow \quad f_n(x) \in S$.

Note $d_\infty(f_n, f) = \max_{x\in[0,1]} |f_n - f|(x)$

$$\leq 1+\tfrac{1}{n} - 1 = \tfrac{1}{n} \to 0 \quad \text{as } n\to\infty$$

$\therefore \quad f \in \overline{S}$ as $f_n \to f$.

Hence $C \subset \overline{S}$. $\divideontimes$

(2) Claim: $S^{o} = \{f \in X : 1 < f(x) < 5, \ \forall x \in [0,1]\}$.

(Pf: Ex!)

[The topic on compactness of my old notes from 2016-17 are removed from the curriculum as it is overlapped with the topology course. It is replaced by the topic "Elementary Inequalities for Functions" as supplement to the examples of function spaces.]