§1.5 Mean Convergence of Fourier Series

<u>Notation</u> : $R[-\pi,\pi] = \{$ set of Riemann integrable (real) functions on $[-\pi,\pi]$ $\}$

<u>Def</u> (1) $\forall\ f, g \in R[-\pi,\pi]$, the <u>$L^2$-product</u> (<u>$L^2$ inner product</u>) is given by

$$\boxed{\langle f, g\rangle_2 = \int_{-\pi}^{\pi} f(x)g(x)\,dx}$$

(Note: for cpx functions $\langle f, g\rangle_2 = \int_{-\pi}^{\pi} f\bar{g}$)

(2) The <u>L^2-norm</u> of $f \in R[-\pi,\pi]$ is $\|f\|_2 = \sqrt{\langle f, f\rangle_2}$

(3) The <u>L^2-distance</u> between $f, g \in R[-\pi,\pi]$ is

$$\|f-g\|_2$$

(4) We said that <u>$f_n \to f$ in L^2-sense</u> if

$$\|f_n - f\|_2 \to 0 \quad \text{as } n \to \infty .$$

(i.e. $\lim_{n\to\infty} \int_{-\pi}^{\pi} (f_n - f)^2\,dx = 0$, "mean convergence")

Caution: L^2-norm & L^2-distance on $R[-\pi,\pi]$ are not really "norm" & "distance" in strict sense

as $\begin{cases} \|f\|_2 = 0 \not\Rightarrow f = 0 \text{ in } R[-\pi,\pi] \\ \|f-g\|_2 = 0 \not\Rightarrow f = g \text{ in } R[-\pi,\pi]. \end{cases}$

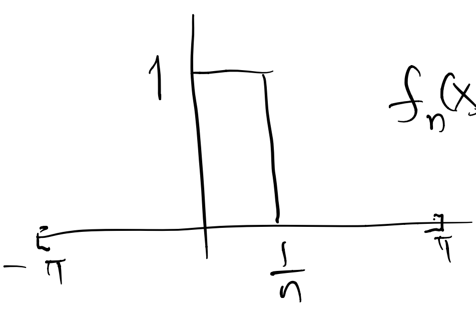
(We only have $\begin{cases} f=0 \text{ almost everywhere.} \\ f=g \text{ almost everywhere.} \end{cases}$)

Note: It is not hard to show that $f_n \to f$ uniformly

$\Rightarrow \|f_n - f\|_2 \to 0$.

However $\|f_n - f\|_2 \to 0 \not\Rightarrow f_n \to f$ uniformly

eg



$f_n(x) = \begin{cases} 1 & x \in [0, \frac{1}{n}] \\ 0 & \text{otherwise}. \end{cases}$

Then $\|f_n\|_2^2 = \int_{-\pi}^{\pi} f_n^2 = \frac{1}{n} \to 0 \quad \therefore f_n \to 0$ in L^2-sense

But $f_n \not\to 0$ uniformly. In fact $f_n(x) \to \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{otherwise}. \end{cases}$

(not even pointwise to 0)

Application to Fourier series:

Consider the functions on $[-\pi, \pi]$

$$\begin{cases} \varphi_0 = \frac{1}{\sqrt{2\pi}} & \text{(const. function)} \\ \varphi_n = \frac{1}{\sqrt{\pi}} \cos nx & (n \geq 1) \\ \psi_n = \frac{1}{\sqrt{\pi}} \sin nx & \end{cases}$$

Then

$$\begin{cases} \langle \varphi_m, \varphi_n \rangle_2 = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \\ \langle \varphi_m, \psi_n \rangle_2 = 0 \quad \forall m, n \\ \langle \psi_m, \psi_n \rangle_2 = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \end{cases}$$

$\therefore \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx \right\}_{n=1}^{\infty}$ can be regarded as an "orthonormal basis" in $R[-\pi, \pi]$.

<u>Notation</u> We denote

$$E_N \overset{\text{def}}{=\!=} \operatorname{span}\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin nx\right\}_{n=1}^{N}$$

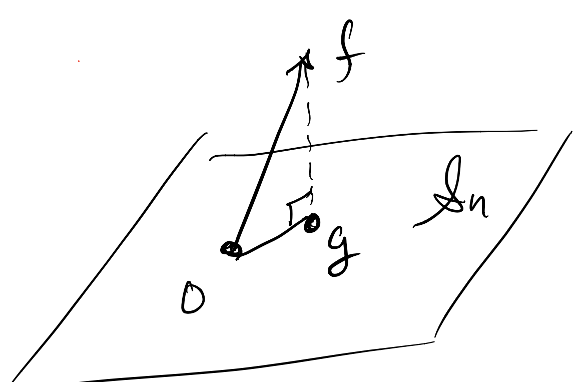
= $(2N+1)$ dim'l vector subspace of $R[-\pi,\pi]$ spanned by the 1^{st} $(2N+1)$ trigonometric functions.

$(\dim E_N = 2N+1)$

In general, if we have an <u>orthonormal set</u> (a <u>orthonormal family</u>) $\{\phi_n\}_{n=1}^{\infty}$ in $R[-\pi,\pi]$ $(\langle \phi_n, \phi_m\rangle_2 = \delta_{mn})$

we set $\mathcal{S}_n = \operatorname{span}\langle \phi_1, \cdots, \phi_n \rangle$

= n-dim'l subspace spanned by the 1^{st} n functions in the orthonormal set.



Then $\forall f \in R[-\pi,\pi]$, we consider the <u>minimization problem</u>

$$\inf\{\|f-g\|_2 : g \in \mathcal{S}_n\}.$$

$\boxed{\begin{array}{l}\underline{\text{Prop 1.14}}\ \text{The unique minimizer of } \inf_{g\in S_n}\|f-g\|_2 \text{ is} \\ \text{attained at the function} \\ \qquad g=\sum_{k=1}^{n}\langle f,\phi_k\rangle_2\,\phi_k \in S_n\end{array}}$

$\underline{\text{Pf}}$: Note that minimize $\|f-g\|_2 \Leftrightarrow \|f-g\|_2^2$ minimize

Then $\forall g\in S_n$, $g=\sum_{k=1}^{n}\beta_k\phi_k$ &

$$\|f-g\|_2^2=\int_{-\pi}^{\pi}\left|f-\sum_{k=1}^{n}\beta_k\phi_k\right|^2 \underset{\text{regarded}}{=\!=} \underset{\Phi(\beta)}{\overset{\shortparallel}{\Phi(\beta_1,\cdots,\beta_n)}}$$

We first need to show that $\Phi(\beta_1,\cdots,\beta_n)\to\infty$ as $\|\beta\|\to+\infty$
$$\|\beta\| = \left(\sqrt{\beta_1^2+\cdots+\beta_n^2}\right)$$

$$\Phi(\beta)=\int_{-\pi}^{\pi}\left(f-\sum_{k=1}^{n}\beta_k\phi_k\right)^2$$

$$=\left(\int_{-\pi}^{\pi}f^2\right)-2\sum_{k=1}^{n}\left(\frac{\beta_k}{\sqrt{2}}\right)\left(\sqrt{2}\langle f,\phi_k\rangle_2\right)+\sum_{k=1}^{n}\beta_k^2$$

$$(2ab\le a^2+b^2)\ \ge\left(\int_{-\pi}^{\pi}f^2\right)-\sum_{k=1}^{n}\left(\frac{\beta_k^2}{2}+2\langle f,\phi_k\rangle_2^2\right)+\sum_{k=1}^{n}\beta_k^2$$

$$= \left(\int_{-\pi}^{\pi} f^2\right) - 2\sum_{k=1}^{n} \langle f, \phi_k\rangle_2^2 + \frac{1}{2}\sum_{k=1}^{n} \beta_k^2$$

$$\longrightarrow +\infty \quad \text{as } \|\beta\| = \left(\sum_{k=1}^{n} \beta_k^2\right)^{1/2} \to +\infty .$$

$\therefore$ $\Phi(\beta)$ attains a minimum at some finite point $\beta = (\beta_1, \cdots, \beta_n)$.

Then easy calculus $\Rightarrow$ the unique minimum is given by

$$\beta_k = \langle f, \phi_k\rangle_2, \ \forall\, k = 1, \cdots, n$$

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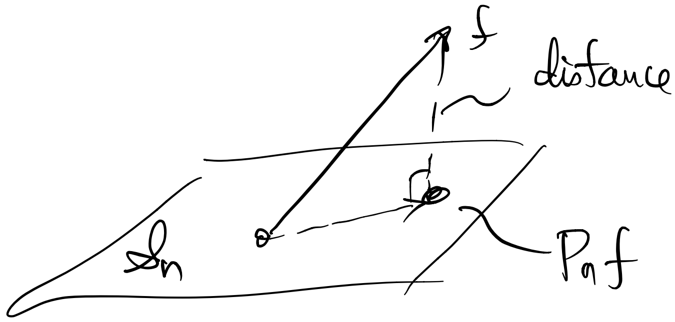
Notes: (1) The minimizer $g = \sum_{k=1}^{n} \langle f, \phi_k\rangle_2 \phi_k$ of $\|f - g\|_2$ over S_n is called the orthogonal projection of f on S_n & denoted by $P_n f$.

(2) $\operatorname{dist}(f, S_n) \left(= \inf\{\operatorname{dist}(f,g) : g \in S_n\}\right)$

$= \|f - P_n f\|_2$



Cor 1.15 For 2π-periodic function f integrable on $[-\pi, \pi]$ and $n \geq 1$,

$$\|f - S_n f\|_2 \leq \|f - g\|_2 \quad \forall\, g \text{ of the form}$$

$$g = \alpha_0 + \sum_{k=1}^{n} (\alpha_k \cos kx + \beta_k \sin kx)$$

with $\alpha_0, \alpha_k, \beta_k \in \mathbb{R}$

($S_n f$ = nth partial sum of the Fourier Series of f)

Pf: By def. of Fourier coefficients $S_n f = P_n f$ of the span $\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos kx, \frac{1}{\sqrt{\pi}}\sin kx\right\}_{k=1}^{n}$:

$$\begin{cases} a_0 = \langle f, \frac{1}{\sqrt{2\pi}} \rangle_2 \cdot \frac{1}{\sqrt{2\pi}} \\ a_n \cos nx = \langle f, \frac{1}{\sqrt{\pi}} \cos nx \rangle_2 \cdot \frac{1}{\sqrt{\pi}} \cos nx \\ b_n \sin nx = \langle f, \frac{1}{\sqrt{\pi}} \sin nx \rangle_2 \cdot \frac{1}{\sqrt{\pi}} \sin nx \end{cases} \quad (\text{Ex!})$$

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