The Power and "Way" of Mathematics

Part II

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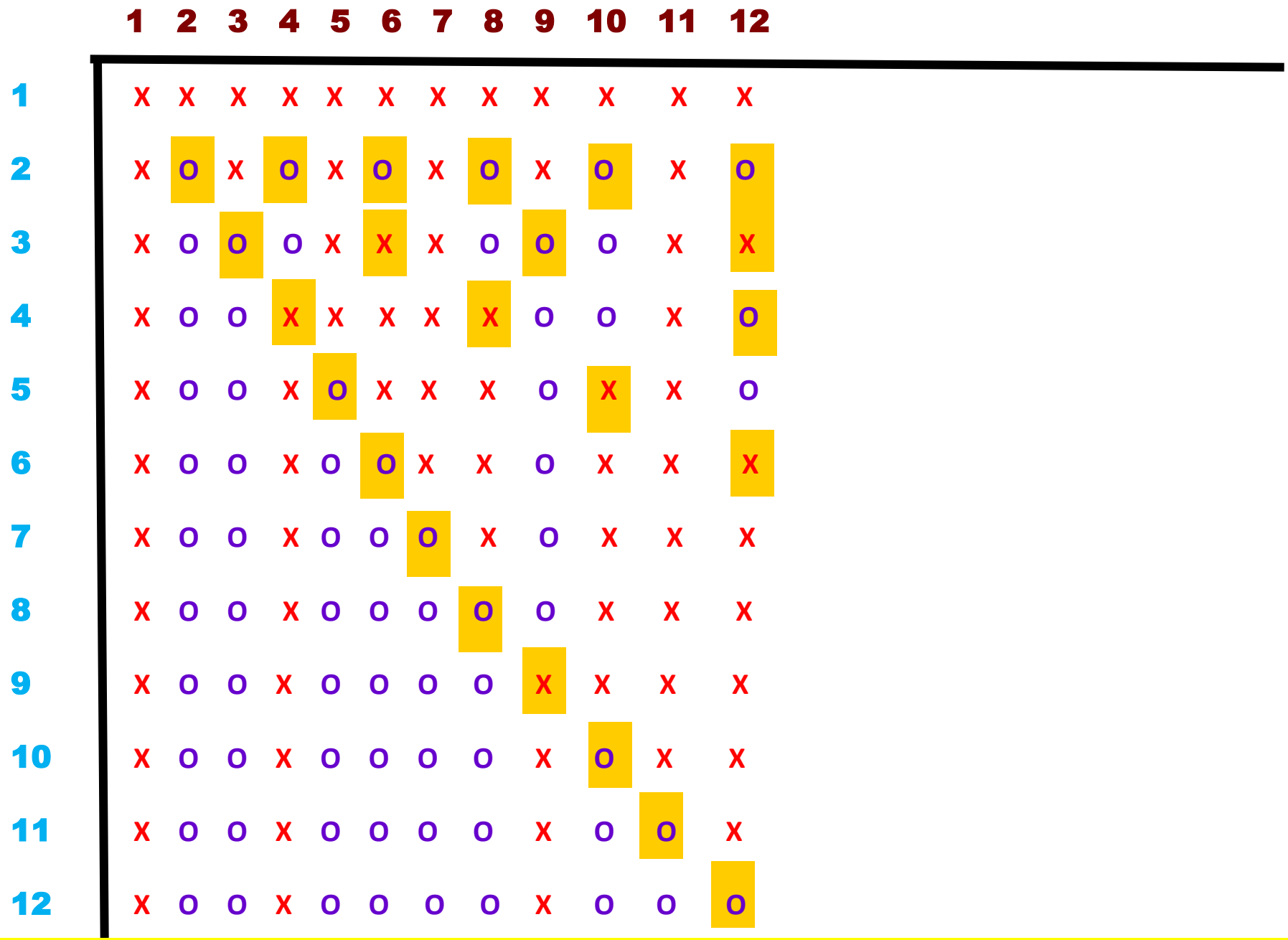
Question :

N persons went to meddle with N closed lockers in a row. The first person opens all lockers. The second person closed every second locker starting with the 2nd one. The third person changed the state (opened or closed) of every third locker starting with the 3rd one, and so on until the Nth person changed the state of the Nth locker.

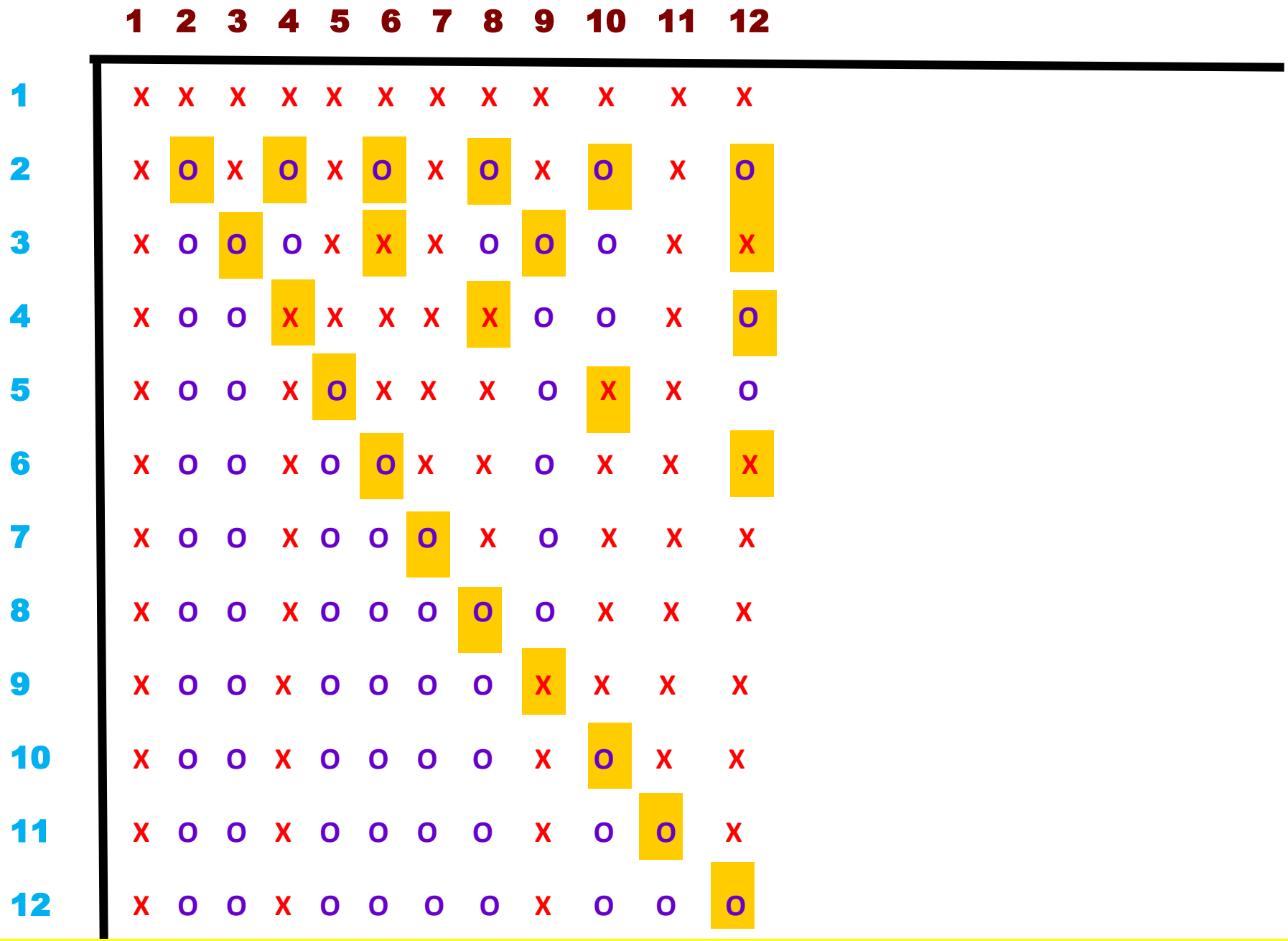
At the end which lockers were left open?

	1	2	3	4	5	6	7	8	9	10	11	12
1	X	X	X	X	X	X	X	X	X	X	X	X
2	X	O	X	O	X	O	X	O	X	O	X	O
3	X	O	O	O	X	X	X	O	O	O	X	X
4	X	O	O	X	X	X	X	X	O	O	X	O
5	X	O	O	X	O	X	X	X	O	X	X	O
6	X	O	O	X	O	O	X	X	O	X	X	X
7	X	O	O	X	O	O	O	X	O	X	X	X
8	X	O	O	X	O	O	O	O	O	X	X	X
9	X	O	O	X	O	O	O	O	X	X	X	X
10	X	O	O	X	O	O	O	O	X	O	X	X
11	X	O	O	X	O	O	O	O	X	O	O	X
12	X	O	O	X	O	O	O	O	X	O	O	O

X = opened O = closed



X ↔ lockers which had undergone an even number of changes



X ↔ numbers with an odd number of factors

Q : Which positive integer *N* has an odd number of factors?

Examples :

Factors of **4** are 1, 2, 4.
Factors of **8** are 1, 2, 4, 8.
Factors of **9** are 1, 3, 9.
Factors of **10** are 1, 2, 5, 10.
Factors of **11** are 1, 11.
Factors of **12** are 1, 2, 3, 4, 6, 12.

If A is a factor of the positive integer N, then N/A is also a factor of N. Note that N/A is not equal to A unless N is a perfect square, because in that case $N/A = A$ implies $N = A^2$.

Thus the factors of N can be paired up with one left-over when and only when N is a perfect square.

Therefore, N has an odd number of factors if and only if N is a perfect square.

Conclusion : All lockers with its number a perfect square will be left open, and all other lockers will be closed.

- **Logical Deduction**
- **Axiomatic (Postulational) Approach**

“The true method must provide us with a *filum Ariadnes*, that is to say a kind of sensitive and coarse means that guides the mind, in the same way as lines drawn in geometry and the type of operations that are prescribed to apprentices in Arithmetic. Without that our mind would not know how to go along a long path without straying.”

G. W. Leibniz

Gottfried Wilhelm Leibniz (1646-1716)

If we could find characters or signs appropriate for expressing all our thoughts as definitely and as exactly as arithmetic expresses numbers or geometric analysis expresses lines, we could in all subjects in so far as they are amenable to reasoning accomplish what is done in Arithmetic and Geometry.

Gottfried Wilhelm Leibniz (1677)

Gottfried Wilhelm Leibniz (1646-1716)

“The design of the following treatise is to investigate the fundamental laws of these operations of the mind by which reasoning is performed; to give expression to them in the symbolic language of a Calculus, and upon this foundation to establish the science of Logic and construct its method …”

George Boole (1815-1864)

George Boole

An Investigation into the Laws of thought (1854)

George Boole, *An Investigation of the Laws of Thought* (1854), Chapter II.

PROPOSITION I

All the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements, viz.:

1st. *Literal symbols as x, y, etc., representing things as subjects of our conceptions.*

2nd. *Signs of operation, as +, −, ×, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the same elements.*

3rd. *The sign of identity,* =.

And these symbols of Logic are in their use subject to definite laws, partly agreeing with and partly differing from the laws of the corresponding symbols in the science of Algebra.

A statement (in mathematics) has a truth value, either T (true) or F (false).

In mathematics we use frequently conditional statements of the form "if A then B".

In mathematical notation we write this as $A \to B$.

If the statement $A \to B$ is true, we write

$$A \Longrightarrow B .$$

The Hatter opened his eyes very wide on hearing this; but all he said was, "Why is a raven like a writing-desk?"

"Come, we shall have some fun now!" thought Alice. "I'm glad they've begun asking riddles. – I believe I can guess that," she added aloud.

"Do you mean that you think you can find out the answer to it?" said the March Hare.

"Exactly so," said Alice.

"Then you should say what you mean," the March Hare went on.

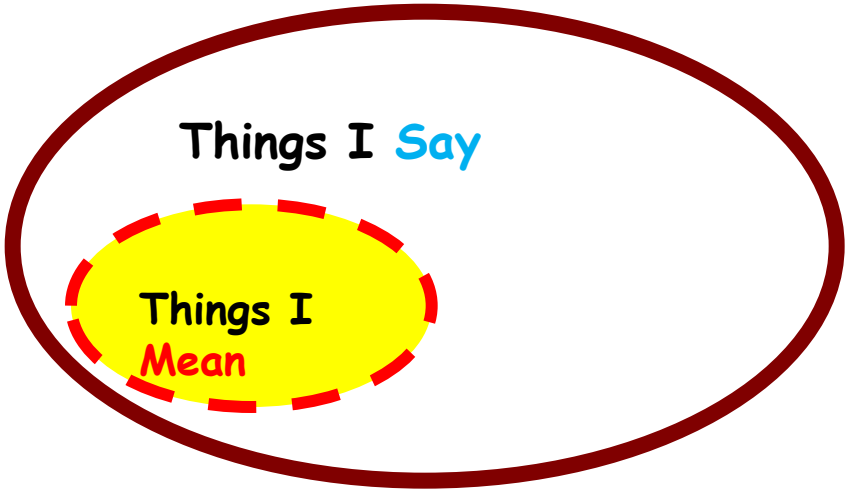
"I do," Alice hastily replied; "at least – at least I mean what I say – that's the same thing, you know."

"Not the same thing a bit!" said the Hatter.

Chapter VII: A Mad Tea Party
Lewis Carroll, *Alice's Adventures in Wonderland* (1865)

If I mean it, then I say it. (I say what I mean.) This conditional statement also tells us that if I do not say it, then I do not mean it. (But if I do not mean it, I may say it or I may not say it.)
We know that there won't be anything which I mean but I do not say.

(But there can be something which I say but I do not mean.)



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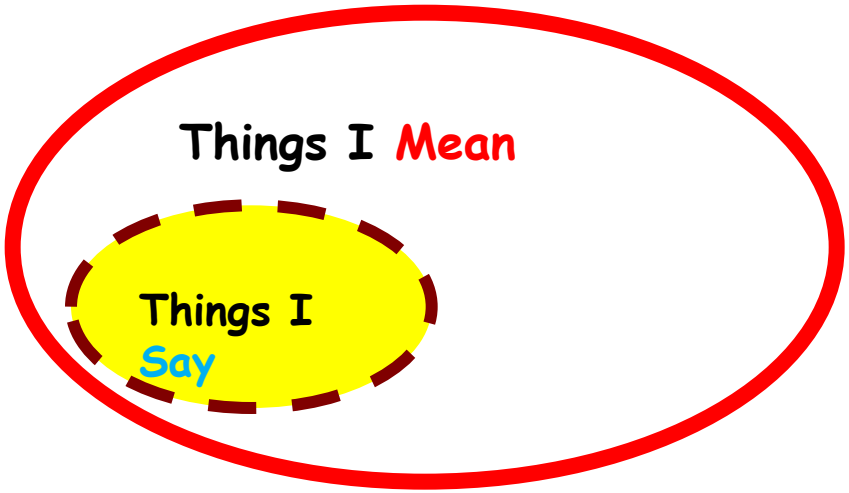


《明報》2004.01.14

Because 因為……所以 Therefore

中學的數學有絕對的因爲和所以，有專用符號∵（因爲）∴（所以），我比較懶，寫其他科目筆記會用三點代替兩個中文字的。

醫學上的因爲和所以要有證據才成立的，譬如病毒性肝炎由肝炎病毒引起，因爲感染肝炎病毒，所以有病毒性肝炎。但原發性高血壓的病因可能是遺傳因素、緊張和壓力、過胖、抽煙、缺乏運動或攝取過多鹽。但當中沒有必然的因果關係，有人有齊上述問題，不一定有高血壓，高血壓患者不一定過胖或抽煙。

關目
關麗珊
www.voy.com/144163/
（留言板）

簡單來說，商業社會充斥許多不是因果關係的謬誤，有爲了宣傳，有爲攻擊對手，有爲無知和反智。好像「因爲人吃飯，所以吃飯的是人」，驟眼看沒有問題，但多想一重就知狗可以吃飯，難道狗吃飯就會變人嗎？

近日看到因爲有子宮頸水瘤，所以變子宮頸癌，或因爲性格X，所以結局Y之類的謬誤。

許多評論的「因爲……所以」玩文字符號，將∵和∴的上下三點轉來轉去，有時倒果爲因，有時因果互不關連，最大問題是學術背景已非「品質保證」。

∵許多人樂意混淆是非黑白，∴灰色成爲最受追捧的顏色。

∵ = because　　∴ = therefore

$A \Rightarrow B$　　because A, (therefore) B

not $A \Rightarrow$ not B

- Because it is an expensive item, it is a good item.
  It is an expensive item, therefore it is a good item.
- Because it is not an expensive item, it is not a good item.
  It is not an expensive item, therefore it is not a good item.
- Because it is not a good item, it is not an expensive item.
  It is not a good item, therefore it is not an expensive item.
- Because it is a good item, it is an expensive item.
  It is a good item, therefore it is an expensive item.

In a conditional statement

"if *p* then *q*"

the statement *p* is called a SUFFICIENT CONDITION of *q*, meaning that *p* is sufficient to guarantee *q*.

The statement *q* is called a NECESSARY CONDITION of *p*, meaning that *q* follows necessarily as a consequence of *p*.

A sufficient condition needs **not** be necessary. A necessary condition needs **not** be sufficient.

Failure in distinguishing the two is a common mistake in reasoning. For instance, the following argument is not valid :

> If I am the President of the CUHK, then you will get an A in this course.
> I am not the President of the CUHK.
> Therefore, you do not get an A in this course.

"I am the President of the CUHK" is a sufficient but may not be a necessary condition for "you get an A in this course". (If you put in the requisite effort, you still get an A, irrespective of my post.)

What we can infer from the first statement is : If you do not get an A, then I am not the President of the CUHK, because otherwise you should get an A!

If we have

$$A \Longrightarrow B$$

and also

$$B \Longrightarrow A \quad ,$$

then we say that

"A if and only if B" **.**

Aristotelian Logic (mid 4th century B.C.) in *Organons*

"a **syllogism** is discourse in which, certain things being stated, something other than what is stated follows of necessity from being so" (*Prior Analytics I*)

A **syllogism** consists of three subject-predicate propositions, two of which are called the **premises** and the third the **conclusion** of the argument.

e.g. All men are mortal
Socrates is a man
Socrates is mortal

S **a** P	ALL S is P.
S **e** P	NO S is P.
S **i** P	SOME S is P.
S **o** P	SOME S is NOT P.

Type I	Type II	Type III	Type IV
M $*$ P	P $*$ **M**	**M** $*$ P	P $*$ **M**
S $*$ **M**	S $*$ **M**	**M** $*$ S	**M** $*$ S
S $*$ P	S $*$ P	S $*$ P	S $*$ P

e.g.

M **a** P
S **a** **M**

S **a** P

M : x is a man
S : x is Socrates
P : x is mortal

Altogether there are $4^4 = 256$ combinations.

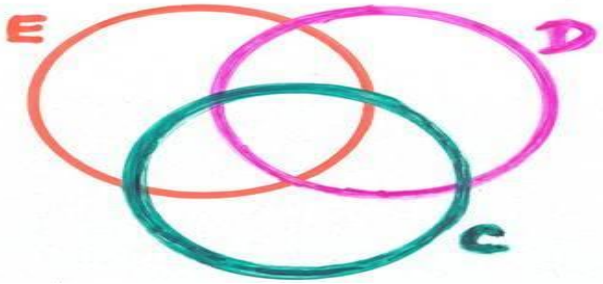
Question: Which of these 256 combinations is a correct argument (a **mood**)?

No emperors are dentists;
All dentists are dreaded by children.
No emperors are dreaded by children.

Lewis Carroll (Charles Lutwidge Dodgson)
Symbolic Logic (1896)

E : x is an emperor
D: x is a dentist
C: x is dreaded by children

No E is D	E e D
All D is C	D a C
No E is C	E e C



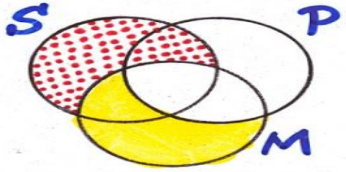
This is **not** a valid argument.

D a C
E e D
E e C

Type I
aee

I

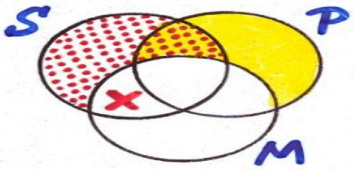
M	a	P	(All M is P.)
S	a	M	(All S is M.)
S	a	P	(All S is P.)



valid argument ✓

II

P	a	M	(All P is M.)
S	a	M	(All S is M.)
S	a	P	(All S is P.)



not valid argument ✗

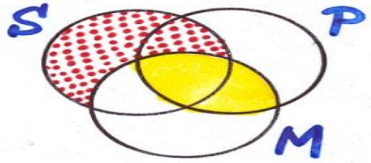
I

M	a	P	(All M is P.)
S	i	M	(Some S is M.)
S	i	P	(Some S is P.)

valid argument ✓

II

P	e	M	(No P is M.)
S	a	M	(All S is M.)
S	e	P	(No S is P.)

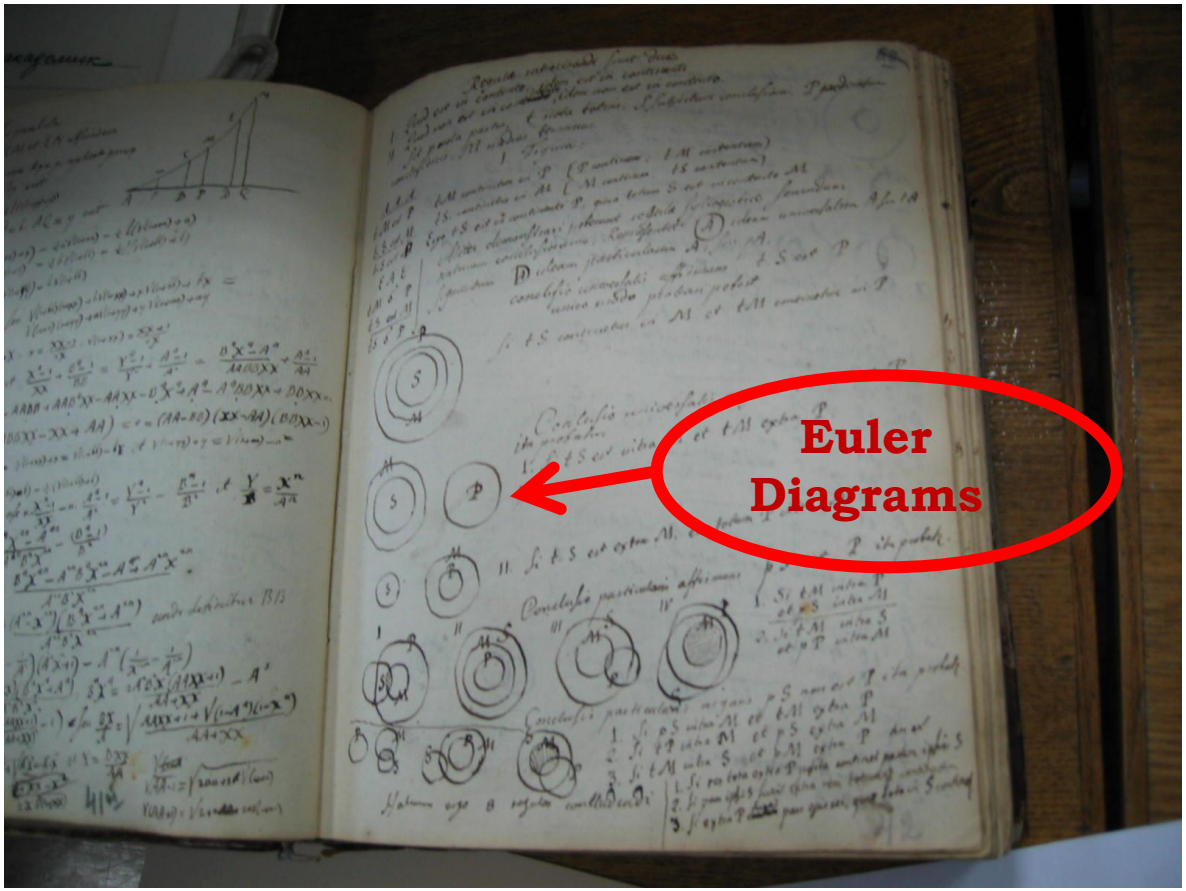


valid argument ✓

Euler diagram (*Lettres à une Princesse d'Allemagne*, 1768)

Barbara Celarent *primae* Darii Ferioque
Cesare Camestres Festino Baroco *secundae*
Tertia *grande* *sonans* *recitat* Darapti Felapton
Disamis Datisi Bocardo Ferison
Quartae *sunt* Bramantip Camenes Dimaris
Fesapo Fresison

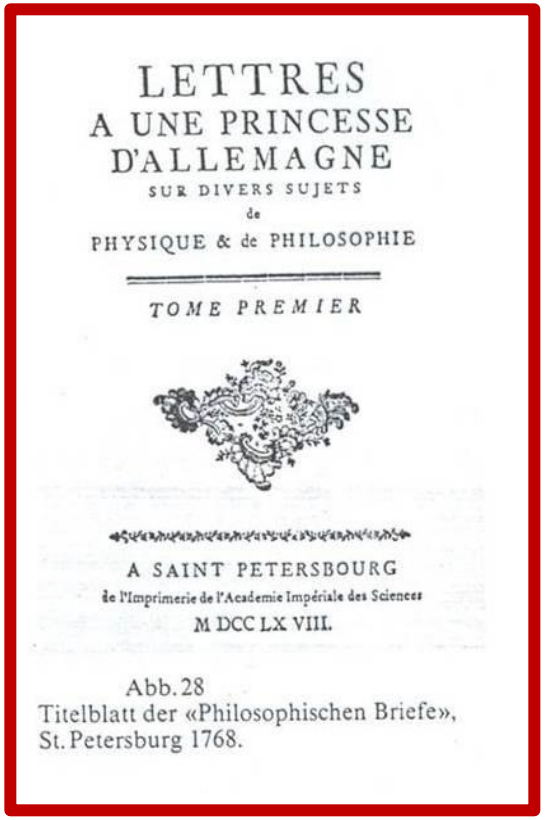
(Peter of Spain, 1210-1277)



One of the many notebooks of Leonhard Euler in the Archive of the Berlin-Brandenburg Academy of Sciences and Humanities

Leonhard Euler (1707-1783)

L. Euler, *Lettres a une princesse d'Allemagne* (1768)



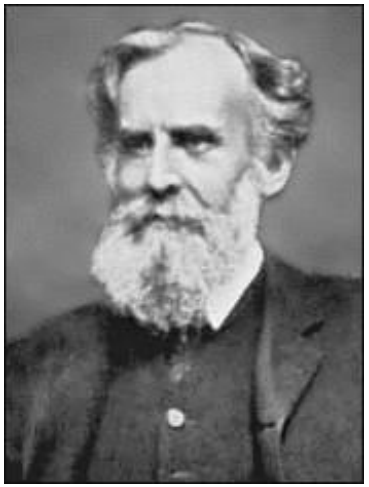
LETTRES
A UNE PRINCESSE
D'ALLEMAGNE
SUR DIVERS SUJETS
de
PHYSIQUE & de PHILOSOPHIE

TOME PREMIER

A SAINT PETERSBOURG
de l'Imprimerie de l'Academie Impériale des Sciences
M DCC LX VIII.

Abb. 28
Titelblatt der «Philosophischen Briefe», St. Petersburg 1768.

A Venn Diagram is a diagram used to depict all possible relationship between a finite collection of different sets.



John Venn
(1834-1923)

A Venn Diagram on a stained window in Gonville & Caius College, Cambridge University.

Lewis Carroll (Charles Lutwidge Dodgson), author of *Alice's Adventures in Wonderland* (1865) also wrote *Symbolic Logic* (1896), in which he gave many exercises. Here are three of them. Tell whether each is a valid argument or not.

(1) No professors are ignorant;
All ignorant people are vain.
[Therefore] No professors are vain.

(2) No birds, except peacocks, are proud of their tails; Some birds, that are proud of their tails, cannot sing.
[Therefore] Some peacocks cannot sing.

(3) All lions are fierce;
Some lions also do not drink coffee.
Hence, some creatures that drink coffee are not fierce.

Lewis Carroll [Charles Lutwidge Dodgson] (1832-1898)

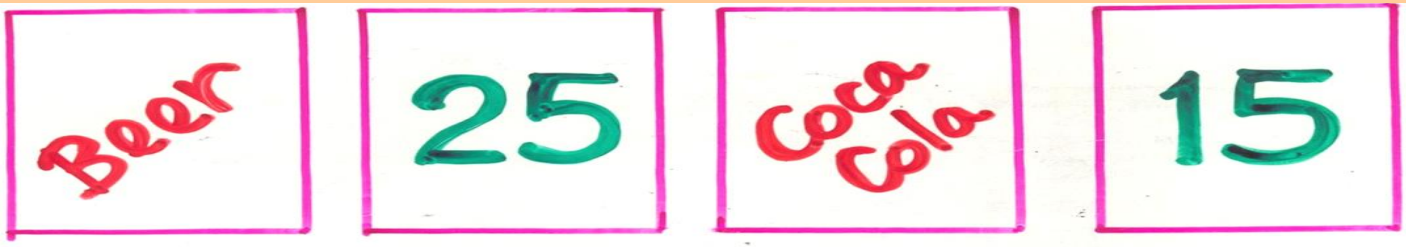
Each card has a number on one side and a letter on the reversed side.

"If a card has A on one side, then it has 4 on the reversed side."

Q. To check the truth of this statement by turning over the *least* number of cards, which cards should you turn over?

P. C. Wason, Reasoning about a rule, Quarterly Journal of Experimental Psychology, 23 (1968), 273-281.

Each card has the age of a person on one side and the beverage that person orders on the reversed side.

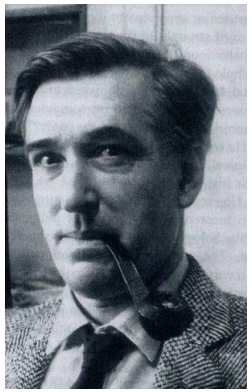


Q. To check whether the following rule is violated or not by turning over the *least* number of cards, which cards should you turn over?

No person under 18 can drink beer on the premises.

The accumulated results collected in my classes for a decade since the beginning of 2000 give 37.9% and 88.4% for a correct answer to the first question and the second question respectively.

This classic experiment was devised by the English psychologist Peter C. Wason of University College of London in the mid-1960s in connection with his well-known study of psychology of reasoning.



Peter Cathcart Wason (1924-2003)

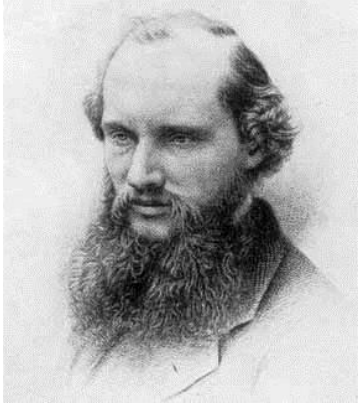
The accumulated results collected in my classes for a decade since the beginning of 2000 give 37.9% and 88.4% for a correct answer to the first question and the second question respectively.

The two questions are actually "isomorphic", requiring exactly the same reasoning, except that the first question is phrased in an abstract mathematically-sounding setting and the second question is phrased in a daily-life setting.

(If somebody is drinking beer or if somebody is less than 18-year-old, we certainly would check. If somebody is drinking coca-cola or if somebody is over 18-year-old, who cares!)

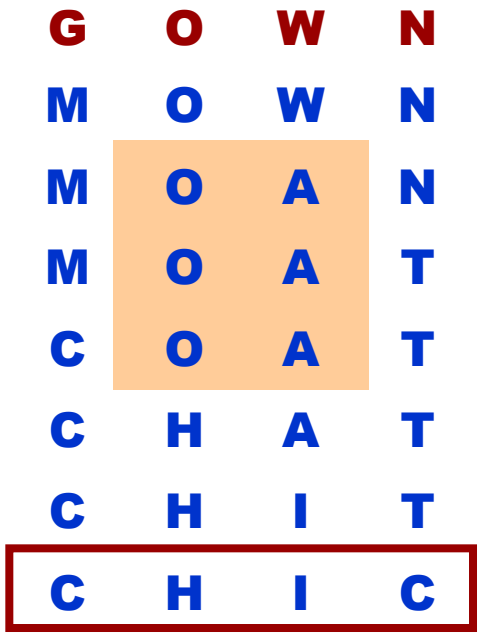
“Do not imagine that mathematics is hard and crabbed, and repulsive to common sense. It is merely the etherealization of common sense.”

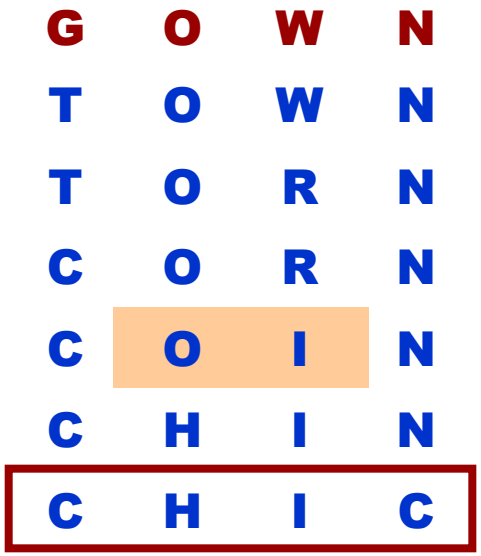
Lord Kelvin
(William Thomson)
1824–1907

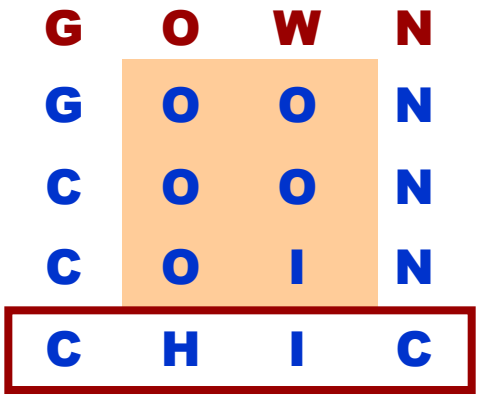


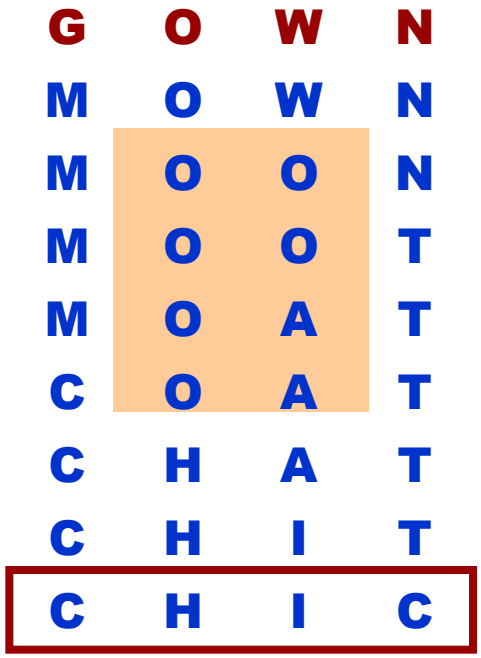
Try to change the word **GOWN** to the word **CHIC** by changing exactly one alphabet at each step (but keeping at each step a word with meaning).

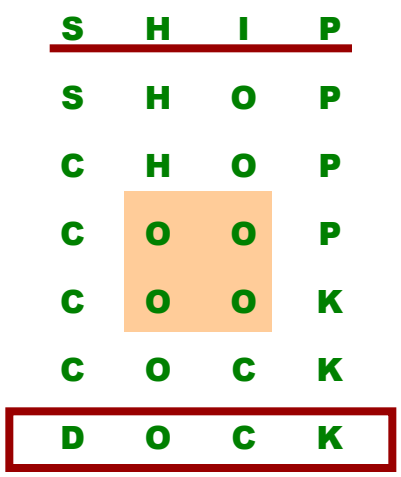
Try the same with the word **SHIP** and the word **DOCK**.

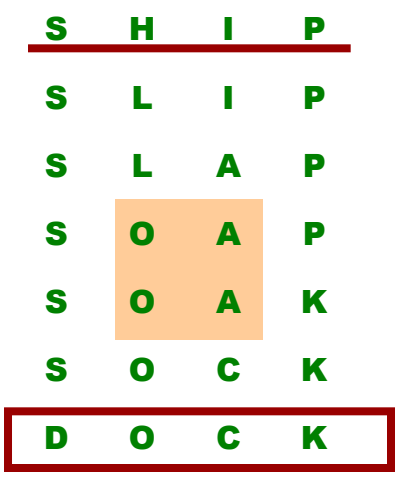

GOWN
MOWN
MOAN
MOAT
COAT
CHAT
CHIT
CHIC

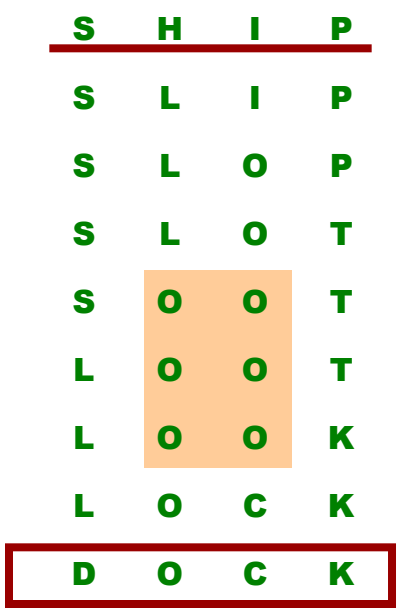

GOWN
TOWN
TORN
CORN
COIN
CHIN
CHIC


GOWN
GOON
COON
COIN
CHIC


GOWN
MOWN
MOON
MOOT
MOAT
COAT
CHAT
CHIT
CHIC


SHIP
SHOP
CHOP
COOP
COOK
COCK
DOCK


SHIP
SLIP
SLAP
SOAP
SOAK
SOCK
DOCK


SHIP
SLIP
SLOP
SLOT
SOOT
LOOT
LOOK
LOCK
DOCK

Theorem :

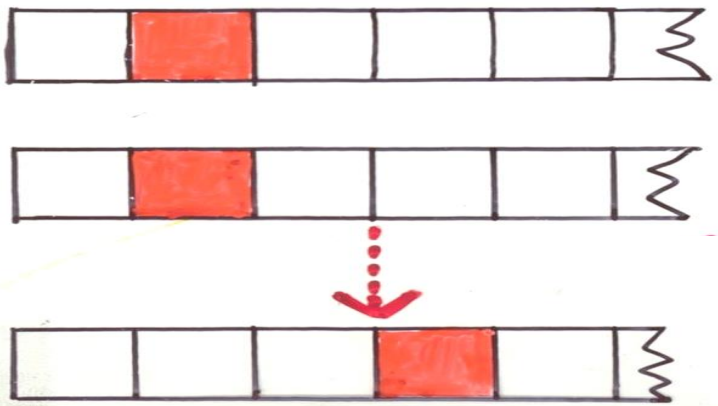
At some point during the change at least two vowels appear.

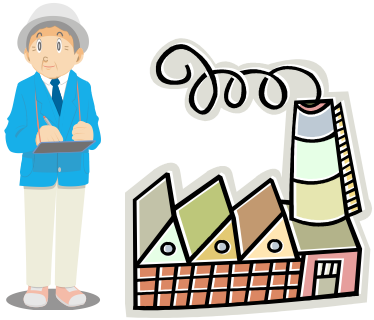
Axiom : Every word has at least one vowel.

Conditions :

(1) Starting word and final word each has exactly one vowel but in different positions.

(2) Each time only one and only one letter is changed.





A

B

C

D

A + B profit of $100 m.

A + B + C profit of $500 m

A + B + D profit of $500 m

A + B + C + D profit of $2000 m

No A , there is no factory.

No B , the factory cannot operate.

How should the profit of $2000m be shared fairly?

A : I own the factory. If there is no factory, there is no job and no profit. Hence I should take $1000m, *B* takes $500m, and *C* , *D* each takes $250m.

B : That is not fair! The factory cannot operate without me. I should get at least as much as *A*. Split the $1500m between us (*A* and me), each getting $750m. *C* and *D* each gets $250m.

C : That is not fair! I earn for *A* and *B* an extra $400m. I should take $400m, and so should *D*. *A* and *B* can split the remaining sum, so each gets $600m.

D : No, we earn for them an extra $1900m. Each of us (*C* and me) should take $950m. Let *A* and *B* split the remaining sum, so each gets $50m.

B : Come on, you do not have your job if there is no factory. You cannot operate without me either. We should at least share that extra $1900m equally among ourselves, each getting $475m. Then *A* and I each gets an extra $50m, making $525m.

C : Why don't we simply divide $2000m equally among ourselves, each getting $500m?

A : That is not fair, because I own the factory!

A	1000	750	600	50	525	500	$591\frac{2}{3}$
B	500	750	600	50	525	500	$591\frac{2}{3}$
C	250	250	400	950	475	500	$408\frac{1}{3}$
D	250	250	400	950	475	500	$408\frac{1}{3}$

In the table the figure 1 means 1 million.

Which allocation would you think is more fair?

John von Neumann Theory Prize
1981

Lloyd Stowell Shapley (1923-2016)

Nobel Memorial Prize in Economic Sciences 2012

Shapley value

Lloyd S. Shapley, A value for *n* – person games, in *Contributions to the Theory of Games*, *Volume 2*, edited by H. Kuhn and A. W. Tucker, 1953.

Basic Rules to be Agreed Upon

(1) **Reward According to Contribution**

The more (less) a person contributes, that person should get more (less).

(2) **No Favouritism**

Two persons with the same contribution will be rewarded the same, irrespective of who those two persons are.

These two rules are reasonable and readily acceptable. The difficult part lies in how much should the reward be distributed (quantitatively).

Confining our attention to what is given and putting aside complications in reality, we can settled on the basic idea of breaking down the enterprise into separate parts, each of which accounts for the contribution from one possible subset of persons involved in the enterprise, then rewarding each accordingly.

We then make use of symbolism and calculate with formulas to reflect the qualitative argument and to make it more quantitative.

Don’t be frightened by the symbolism and calculation, which are just ways of mathematical articulation to record in a precise manner our common sense, and allows us to treat general problems of this kind. Instead of the full detailed technical calculation we are more interested in the choice of criteria and the rationale in adopting them, and in the manner how the answer is forced upon us by the (few) rules we set for ourselves.

Mathematically speaking we devise a function w defined on the set of subsets of $\{A,B,C,D\}$ (which gives the profit when the persons in that subset are involved) and a localized version w_T which focuses on the contribution of different subsets T.

For instances,

$w(\{A\}) = 0$, but $w(\{A, B\}) = 100$,

$w(\{A, B, C, D\}) = 2000$, etc.

$w_T(U) = 1$ if all persons in T are in U, and $w_T(U) = 0$ if not all persons in T are in U. For instance, $w_{\{A,C\}}(\{A, B, C\}) = 1$, but $w_{\{A,C\}}(\{A, B\}) = 0$. The function w_T can be regarded as an enterprise involving at least all persons in T with net profit 1 and 0 otherwise.

We make a simple assumption with conviction merely lying in its simplicity! [***Okham's razor*** : Plurality should not posited without necessity (14^{th} century)] that w is a "linear combination" of those w_T's.

One should interpret the equality in a more concrete sense by applying it on different subsets. The outcome is a system of 15 linear equations in 15 unknowns.

By standard and mechanical means (details of which do not concern us in this lecture) we can solve the system and obtain the function w in terms of the w_T's. Finally we devise a suitable profit function φ_x where x is A, B, C or D. By the criterion that the reward should be proportional to contribution we solve for these φ_x's.

(Shapley Value)

蕭文強, 《數學證明》[*Mathematical Proofs*], 江蘇教育出版社, 1989; corrected edition, 九章出版社, 2007; 大連理工大學出版社, 2008; 2016.

What is the main role of a PROOF?

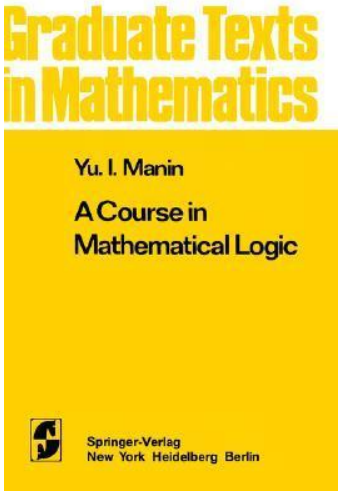
- "ritual" of the trade?
- for the purpose of verification?
- to guard against error?
- for enhancement of understanding?

Main function of a Proof :
To enhance UNDERSTANDING

"A good proof is one which makes us wiser."

Yu I. Manin
A Course in Mathematical Logic
**(English translation by N. Koblitz, 1977;
second edition, 2009, Springer-Verlag)**

Yurii Ivanovich Manin
(1937 -)



Q. Can $1+1141N^2$ be a perfect square?

N	$1+1141N^2$	$\sqrt{1+1141N^2}$
1	1142	33.793…
2	4565	67.564…
3	10270	101.341…
4	18257	135.118…
5	28526	168.896…
6	41077	202.674…
⋮	⋮	⋮
30693385322765657197397208		10367823941572239632371252 15
⋮	⋮	⋮

$$1+1141\times(30693385322765657197397208)^2 = (10367823941572239632371252 15)^2$$

This is a proof, but does it make you any wiser than before?

What about …

Does the preceding proof by direct verification makes you any wiser than before?

What about …

Solve $1+1141N^2=x^2$,

or $x^2-1141N^2=1$;

$$(x-\sqrt{1141}\,N)(x+\sqrt{1141}\,N)=1.$$

Hence $\frac{x}{N}>\sqrt{1141}$ and

$$0<\frac{x}{N}-\sqrt{1141}=\frac{1}{(x+\sqrt{1141}\,N)N}$$

$$<\frac{1}{(\sqrt{1141}\,N+\sqrt{1141}\,N)\,N}$$

$$=\frac{1}{2\sqrt{1141}\,N^2}<\frac{1}{2N^2}$$

It may prove useful to try to approximate $\sqrt{1141}$ by fractions.

$$\sqrt{1141}=33+(\sqrt{1141}-33)=33+\frac{1}{\frac{1}{\sqrt{1141}-33}}$$

$$\frac{1}{\sqrt{1141}-33}=\frac{\sqrt{1141}+33}{52}=1+\frac{\sqrt{1141}-19}{52}=1+\frac{1}{\frac{52}{\sqrt{1141}-19}}$$

$$\frac{52}{\sqrt{1141}-19}=\frac{52(\sqrt{1141}+19)}{780}=\frac{\sqrt{1141}+19}{15}=3+\frac{\sqrt{1141}-26}{15}=3+\frac{1}{\frac{15}{\sqrt{1141}-26}}$$

$$\frac{15}{\sqrt{1141}-26}=\frac{15(\sqrt{1141}+26)}{465}=\frac{\sqrt{1141}+26}{31}=1+\frac{\sqrt{1141}-5}{31}=1+\frac{1}{\frac{31}{\sqrt{1141}-5}}$$

etc.

$$\sqrt{1141}=33+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cdots}}}$$

$\frac{33}{1}$, $33+\frac{1}{1}=\frac{34}{1}$, $33+\cfrac{1}{1+\frac{1}{3}}=\frac{135}{4}$,

$33+\cfrac{1}{1+\cfrac{1}{3+1}}=\frac{169}{5}$, etc.

Pell's Equation

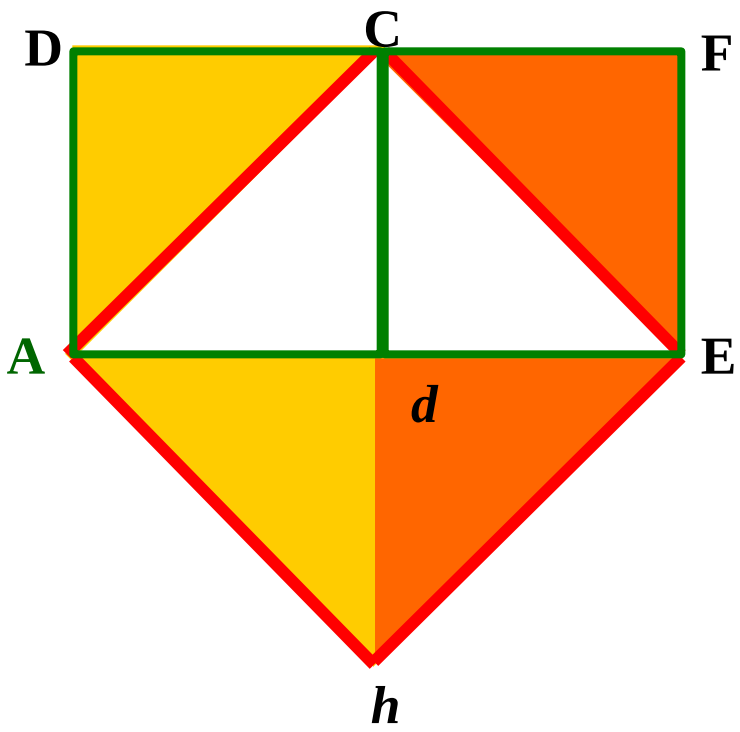
(a misnomer due to Euler!)

Interesting long history of Pell's Equation since 400 B. C. E.

See: https://en.wikipedia.org/wiki/Pell%27s_equation

A.C. Clairaut, *Eléments de géométrie* (1741; 1753)

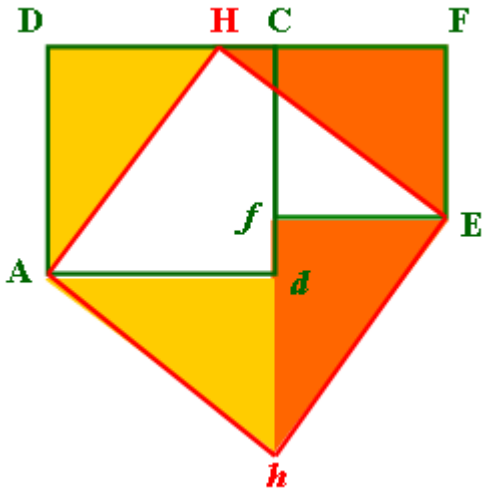
XVI **To make a square equal in area to two equal (smaller) squares.**



ACE*h* is a square

ACE*h* = ADC*d* + CFE*d*

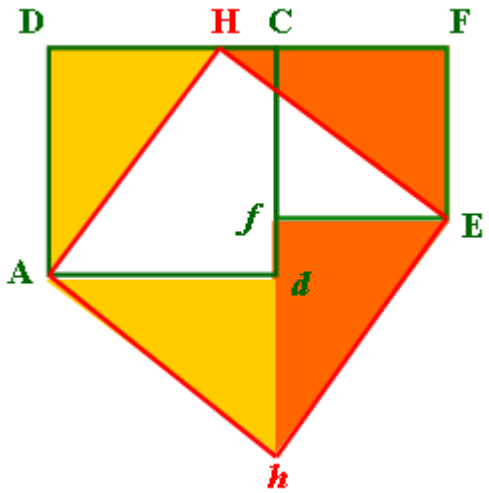
XVII To make a square equal in area to two other taken together.



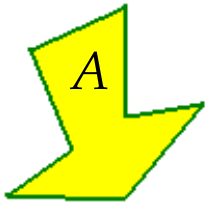
Following the trend of thought in XVI we try to find a point H on DF such that
(i) when ADH is turned around A and when EFH is turned around E, they join at a point *h*.
(ii) AH, HE, E*h*, *h*A are equal and perpendicular.

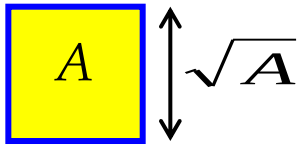
Take H on DF such that DH = CF = EF.

XVIII The square on the hypotenuse of a right triangle is equal to the sum of the squares on the two other sides. (Pythagoras Theorem)



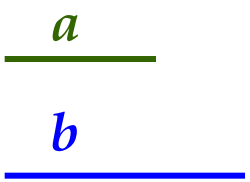
S.D. Agashe, The axiomatic method: Its origin and purpose, *Journal of the Indian Council of Philosophical Research*, vol. 6, no. 3 (1989), 109-118.



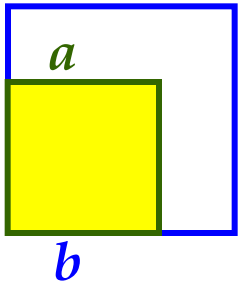


Proposition 14 of Euclid's *Elements Book II*: **To construct a square equal to *a* given rectilineal figure**.

Motivation:



Comparison of two line segments Proposition 3 of *Book I*: Given two unequal straight lines, to cut off from the greater a straight line equal to the less. (relying on **Postulates 1,2,3**)

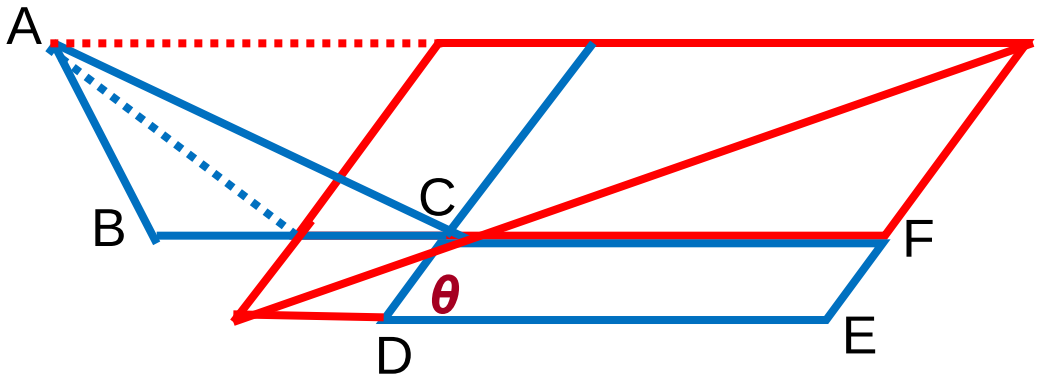


Comparison of two rectilineal figures

(reduce each to a square and compare, relying on **Postulates 4)**

How to construct a rectangle equal (in area) to a given rectilineal figure?

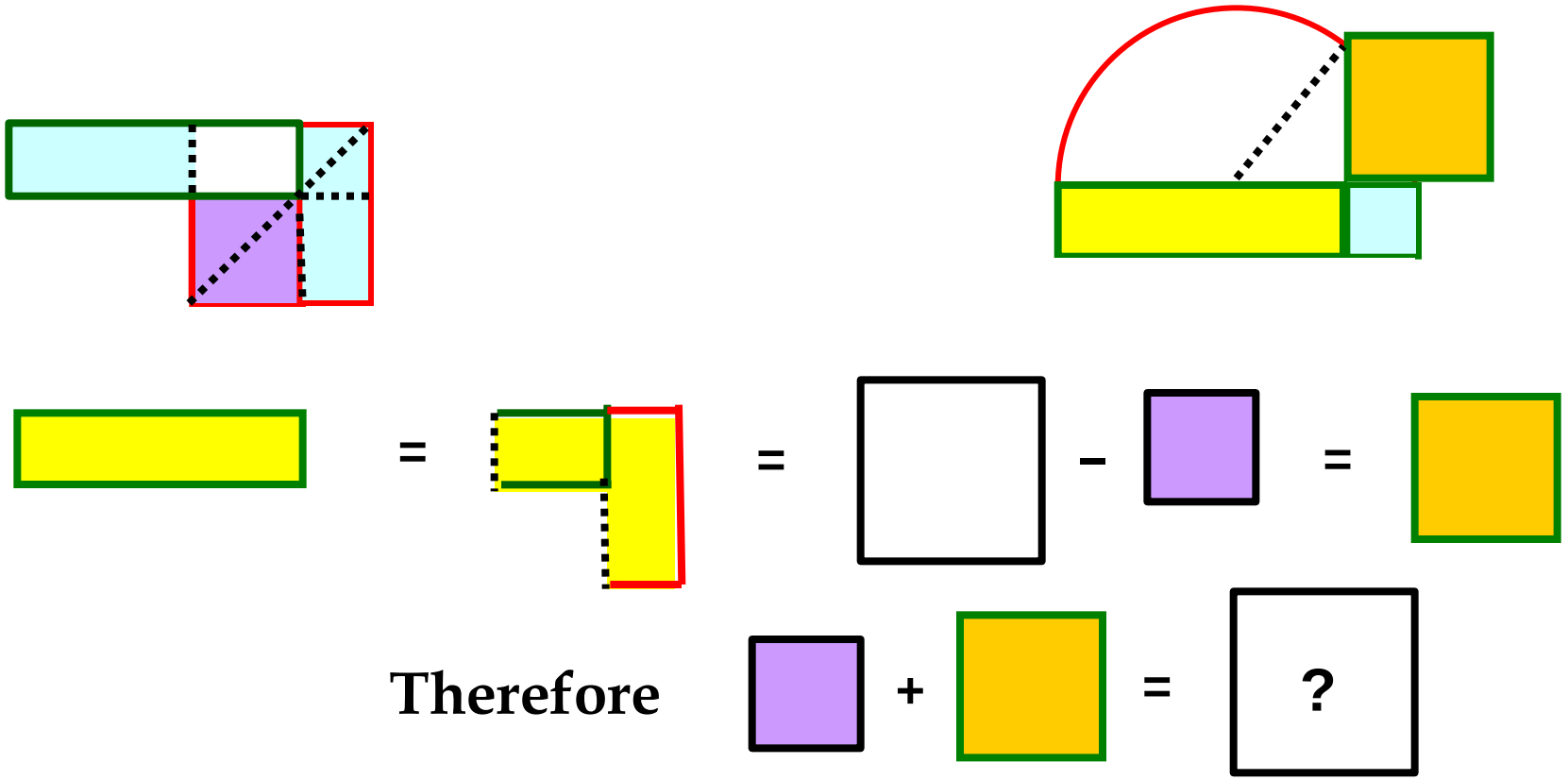
Decompose into triangles and construct a rectangle of one given length (more generally a parallelogram with one given angle and one given length) equal in area to each triangle. (Proposition 42, 44, 45 of *Book I*, relying on **Postulate 5**)



ΔABC = CDEF, CD of given length and angle CDE of given magnitude θ .

How to square a rectangle?

Reduce a rectangle to a gnomon (L-shaped figure).
This is the content of Proposition 5 of *Book II*.



gnomon = difference of two squares

sum of two squares = square? (Proposition 47 of *Book I*)

Pythagoras theorem

Geometry in the sense of **mensuration of figures** was spontaneously developed by many cultures and dates to several millenia B.C.. The science of geometry as we know it, namely, **a collection of abstract statements regarding ideal figures**, the verification of whose validity requires only **pure reason**, was created by the Greeks.

Saul Stahl, *The Poincaré Half-Plane: A Gateway To Modern Geometry*, 1993.

Euclid's *Elements* (c.300 B.C.E.)


Preclarissimus liber elementorum Euclidis perspicacissimi: in artem Geometrie incipit quàfoelicissime:

De principijs p se notis: e primo de diffinitionibus eorundem.

Punctus est cuius ps nō est. ¶ Linea est lōgitudo sine latitudine cui⁹ quidē extremitates sūt duo pūcta. ¶ Linea recta ē ab vno pūcto ad aliū breuissima extēsio ī extremitates suas vtrūq; eox recipiens. ¶ Superficies ē q̄ lōgitudinē ⁊ latitudinē tm̄ habz: cui⁹ termini quidē sūt linee. ¶ Superficies plana ē ab vna linea ad alia extēsio ī extremitates suas recipiēs. ¶ Angulus planus ē duarū lineaꝝ alternus ⁊ tactus: quaꝝ expānsio ē sup superficiē applicatioq; nō directa. ¶ Quādo aūt angulum ⁊inēt due linee recte rectilineū⁹ angulus noiat. ¶ Cūi recta linea sup rectā steterit duoq; anguli vtrobiq; fuerit eqles: eox vterq; rect⁹ erit. ¶ Lineaq; linee superstās ei cui superstat ppendicularis vocat. ¶ Angulus vo qui recto maior ē obtusus dicit. ¶ Angul⁹ vo minor recto acut⁹ appellat. ¶ Termin⁹ ē qd oīs cuiuslibz huius ē. ¶ Figura ē q̄ tmino vel termis ⁊tinet. ¶ Circul⁹ ē figura plana vna q̄dem linea ⁊ētra: q̄ circūferentia noiat: in cui⁹ medio pūct⁹ ē: a quo' oēs linee recte ad circūferētiā exeūtes īdiuicez sūt equales. Et hic quidē pūct⁹ cētrū circuli dr. ¶ Diameter circuli ē linea recta que sup ei⁹ centꝝ trāliens extremitatesq; suas circūferēie applicans circulū ī duo media diuidit. ¶ Semicirculus ē figura plana diametro circuli ⁊ medietate circūferētie ⁊centa. ¶ Portio circuli ē figura plana recta linea ⁊ parte circūferēie ⁊ēta: semicirculo quidē aut maior aut minor. ¶ Rectilinee figure sūt q̄ rectis lineis cōtinent quarū quedā trilatere q̄ trib⁹ rectis lineis: quedā quadrilatere q̄ q̄tuor rectis lineis. q̄dā multilatere que pluribus q̄z quatuor rectis lineis continent. ¶ Figurarū trilaterarū: alia est triangulus hīs tria latera equalia. Alia triangulus duo hīs eqlia latera. Alia triangulus triū inequalium laterū. Rursz iterū alia est ortogoniū: vnū ī. rectum angulum habens. Alia ē ambligonium aliquem obtusum angulum habens. Alia est oxigonium: in qua tres anguli sunt acuti. ¶ Figurarū aūt quadrilaterarū: Alia est ōdratum quod est equilaterū atq; rectangulū. Alia est tetragon⁹ long⁹: q̄ est figura rectangula: sed equilatera non est. Alia est delmuaym: que est equilatera: sed rectangula non est.

Linea
Punctus
Superficies plana.
Circulus
Diameter
Portio maior
minor
Semicirculus
Oxigonius
ambligonius
Tetragō⁹ long⁹
ōdratus
delmuaym

First printed edition of Euclid's *Elements* made in Venice in 1482

13 Books

5 Common Notions

+ 5 Postulates

⇒ 465 Propositions

Its name [μαθηματική] thus makes clear what sort of function this science performs. It arouses our innate knowledge, awakens our intellect, purges our understanding, brings to light the concepts that belong essentially to us, takes away the forgetfulness and ignorance that we have from birth, sets us free from the bonds of unreason; ……

Proclus (ca 410-485)
***A Commentary on the First Book of Euclid's Elements*, Prologue: Part One.**
[Translated by Glenn R. Morrow, 1970]

Curriculum for Higher Education in Plato's Academy

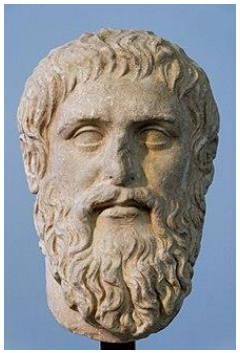
- arithmetic and logistic
- plane geometry
  solid geometry
- astronomy
- harmonics (music theory)

quadrivium

(Boethius, c.480-524)

"When they reach thirty they will be promoted to still higher privileges and tested by the power of Dialectic, to see which can dispense with sight and the other senses and follow truth into the region of pure reality."

Plato: *The Republic*
(c.4th century B.C.)



Plato (*c.* 427 – 347 B.C.E.)

保氏，掌諫王惡，而養國子之道，
乃教之六藝，一曰五禮、二曰六樂、三曰五射、
四曰五馭、五曰六書、六曰九數。
（御）
《周禮・地官司徒下》

Zhou Li [Rites of Zhou] *c.* 2nd century B.C.E.,
probably compiled by the 3rd /4th century B.C.E.

SIX (Gentlemanly) ARTS 六藝

禮 **Rites**
樂 **Music**
射 **Archery**
御 **Charioteering / Horsemanship**
書 **History (Writing)**
數 **Arithmetic (Mathematics)**

The seven liberal arts in the medieval time

QUADRIVIUM = arithmetic, geometry, music, astronomy

TRIVIUM = rhetoric, dialectic, grammar

Histories make men wise;
poets, witty;
the mathematics, subtle;
natural philosophy, deep;
moral, grave;
logic and rhetoric, able to contend.

Of Studies
Francis Bacon

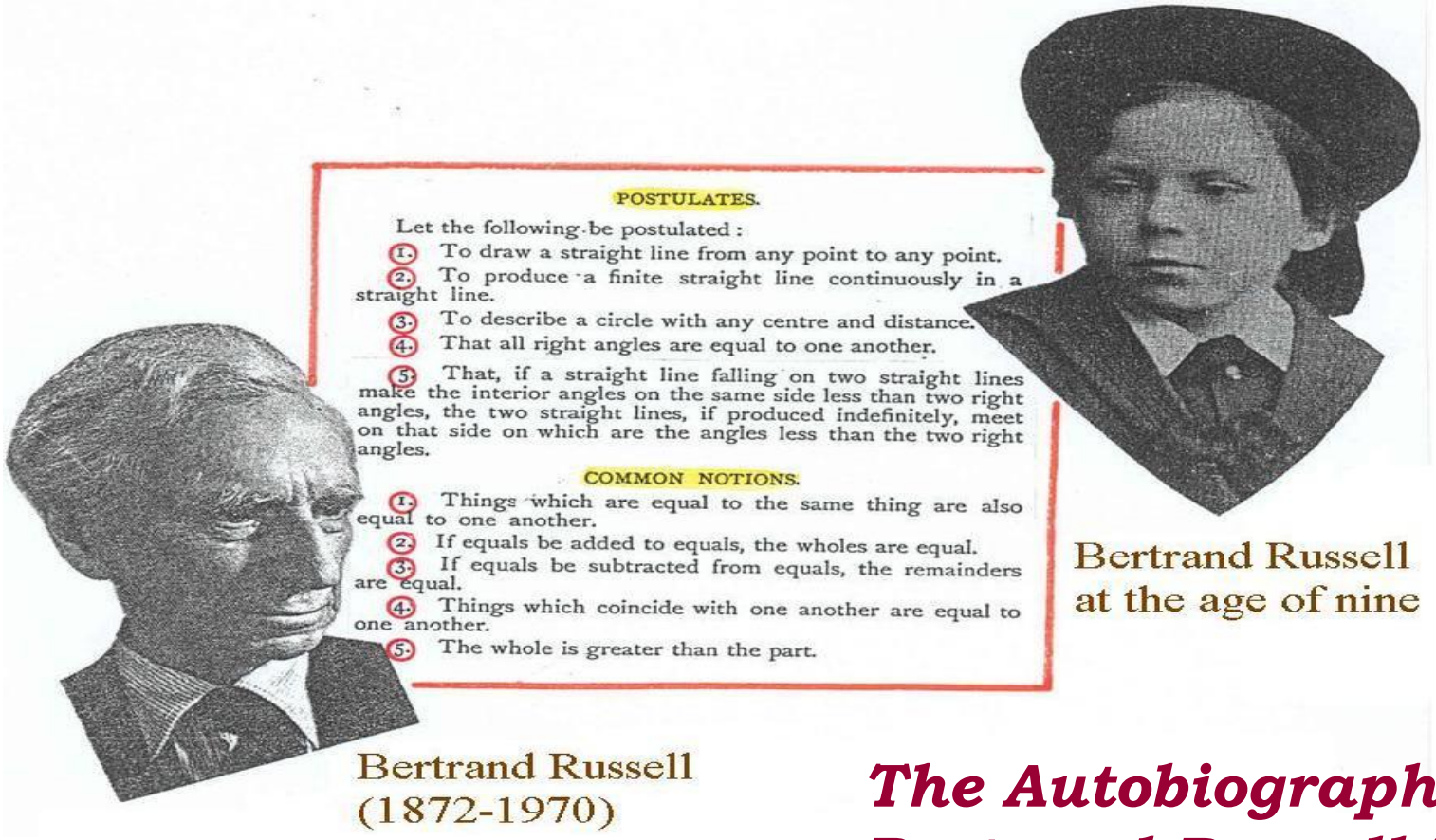
Francis Bacon (1561-1626)

"The spirit of geometry is not only confined to geometry that it cannot be taken out and transferred to other domains of knowledge. A work of morality, politics, criticism, perhaps even eloquence, will become more elegant, other things being equal, if it is touched by the hand of geometry."

Bernard le Bovier de Fontenelle, *Preface sur l'Utilite des Mathematiques et de la Physique* (1729)

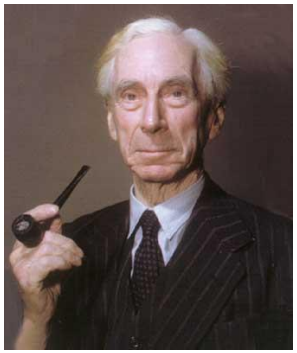
Bernard le Bovier de Fontenelle (1657-1757)

"At the age of eleven, I began Euclid, with my brother as tutor. This was one of the great events of my life, as dazzling as first love."



***The Autobiography of Bertrand Russell* (1967)**

“I had been told that Euclid proved things, and was much disappointed that he started with axioms. At first, I refused to accept them unless my brother could offer me some reason for doing so, but he said, "If you don't accept them, we cannot go on", and as I wished to go on, I reluctantly admitted them *pro temp*.”



Bertrand Russell
(1872-1970)

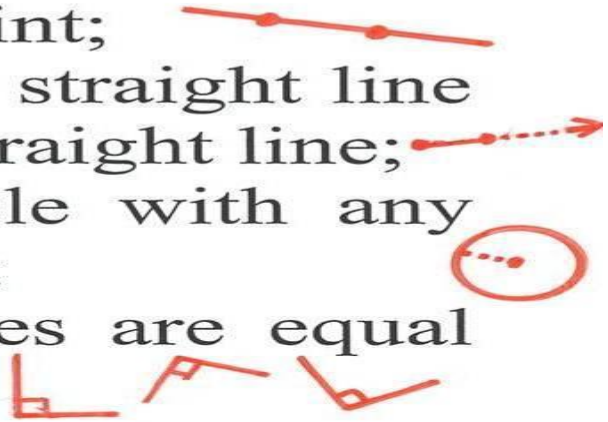
The Autobiography of Bertrand Russell (1967)

Euclid's *Elements* (c.300 B.C.E.)

Postulates in Book I of *Elements*

Let the following be postulated:

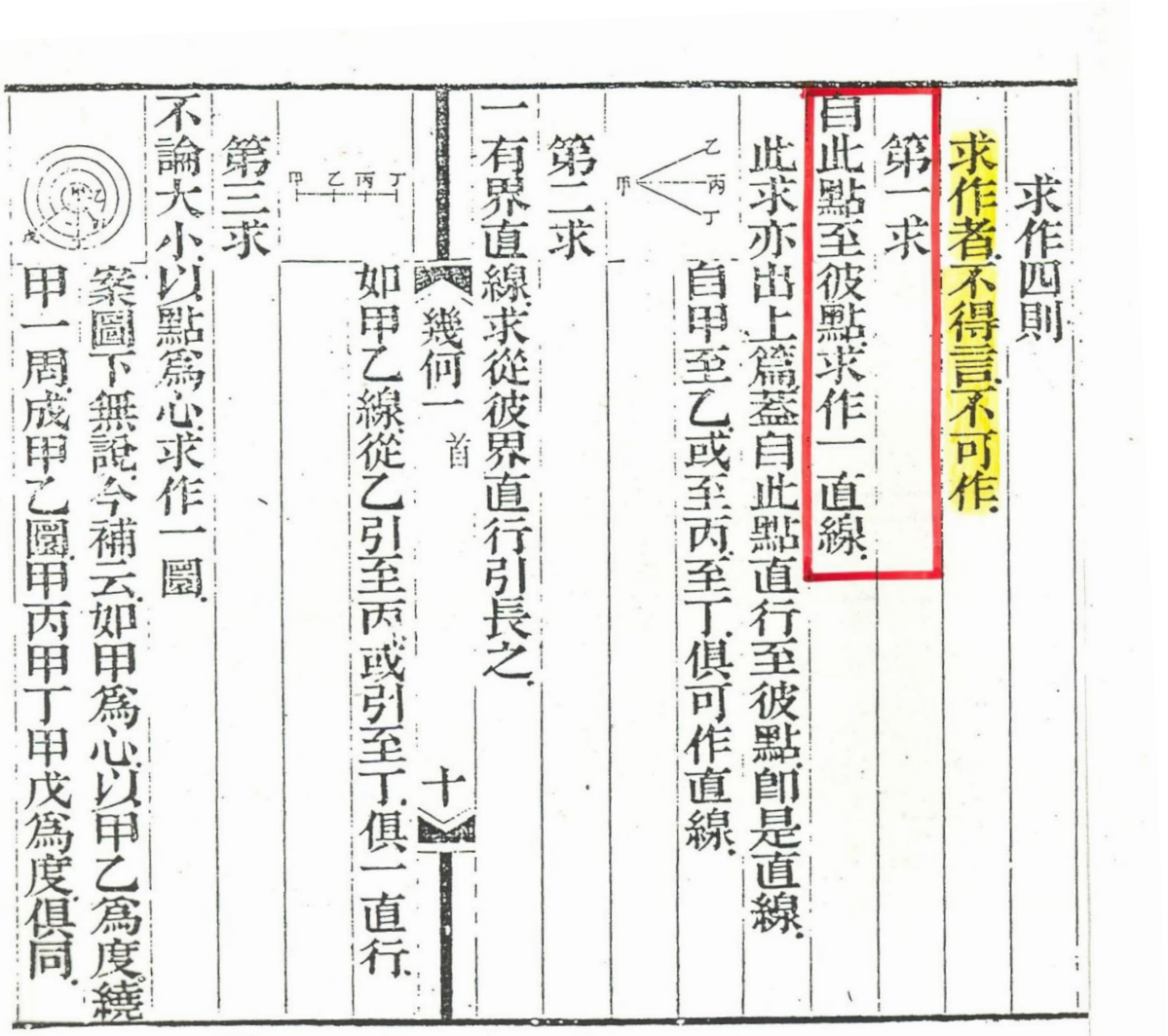
1. To draw a straight line from any point to any point;
2. To produce a finite straight line continuously in a straight line;
3. To describe a circle with any center and distance;
4. That all right angles are equal to one another;
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.



465 Propositions are derived in the thirteen books of *Elements* .

Jihe Yuanben, Book I (1607) 《幾何原本》

[translation of *Elements* by Matteo Ricci and XU Guang-qi]



求作四則

求作者不得言不可作

第一求

自此點至彼點求作一直線

此求亦出上篇蓋自此點直行至彼點即是直線

自甲至乙或至丙至丁俱可作直線

第二求

一有界直線求從彼界直行引長之

如甲乙線從乙引至丙或引至丁俱一直行

第三求

不論大小以點爲心求作一圓

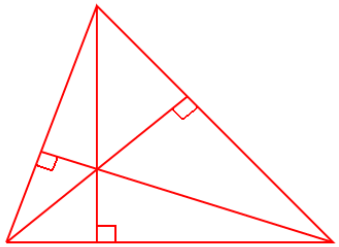
案圖下無說今補云如甲爲心以甲乙爲度繞甲一周成甲乙圓甲丙甲丁甲戊爲度俱同

幾何一 首 十

求作者
不得言不可作
(if it is
requested to
construct this,
it is not
allowed
to say that it
cannot be done)

axioma = making a request

"At the age of twelve I experienced a second wonder of a totally different nature: in a little book(*) dealing with Euclidean plane geometry, which came into my hands at the beginning of a school year."



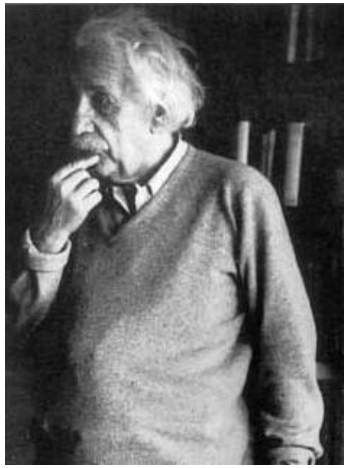
The three altitudes of a triangle are concurrent

(*) **E. Heis, T.J. Eschweiler, Lehrbuch der Geometrie zum Gebrauch an höheren Lehranstalten, Du-Mont & Schauberg, Cologne, 1867 ("holy geometry book")**

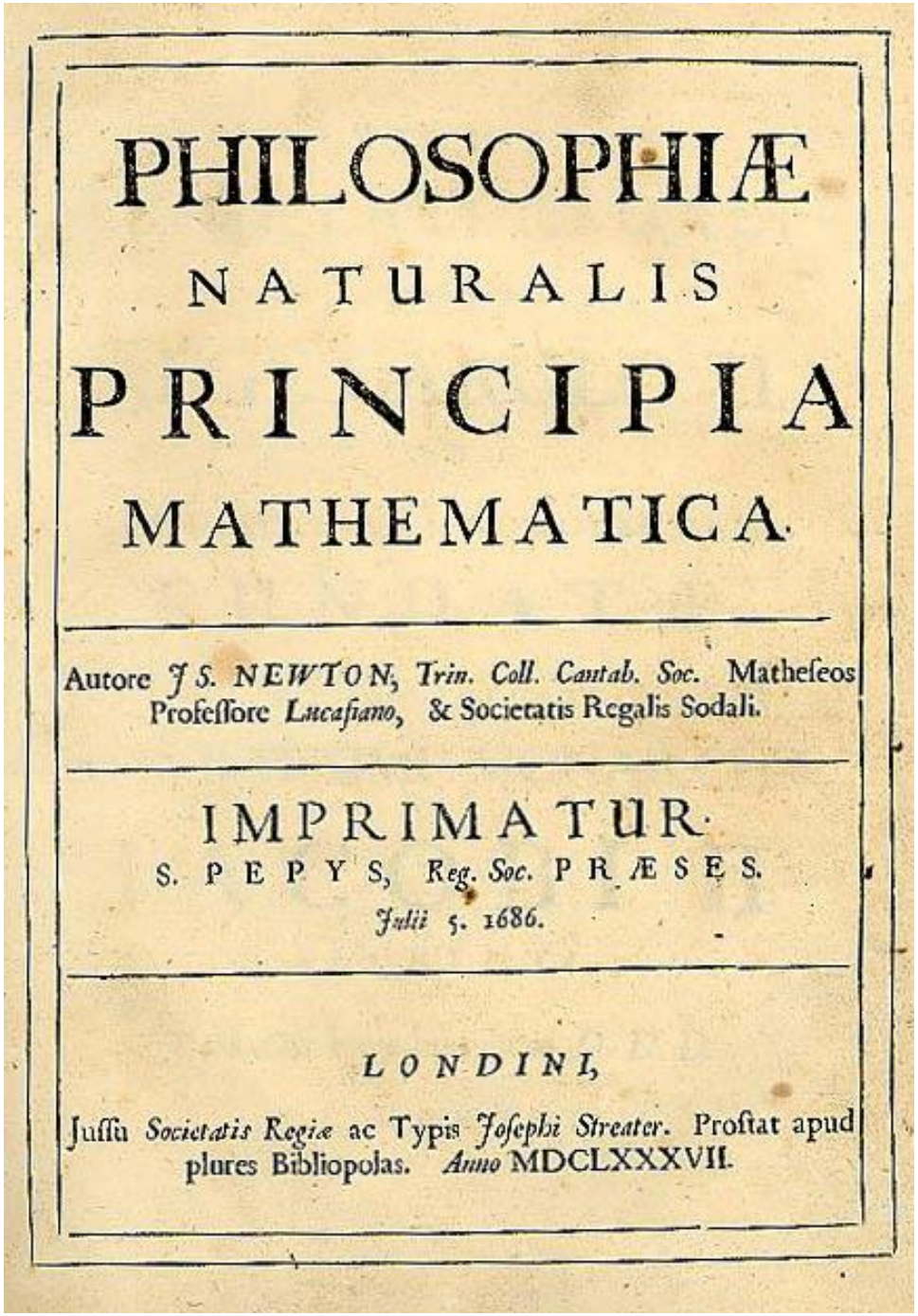
Albert Einstein (autobiographic notes written at the age of 67)

"The lucidity and certainty made an indescribable impression upon me. … it is marvellous enough that man is capable at all to reach such a degree of certainty and purity in pure thinking as the Greeks showed us for the first time to be possible in geometry."

Autobiographic notes (in German) by Albert Einstein, written at the age of 67, in *Albert Einstein: Philosopher-Scientist*, edited by Paul Arthur Schilepp (1949)



Albert Einstein (1879-1955)



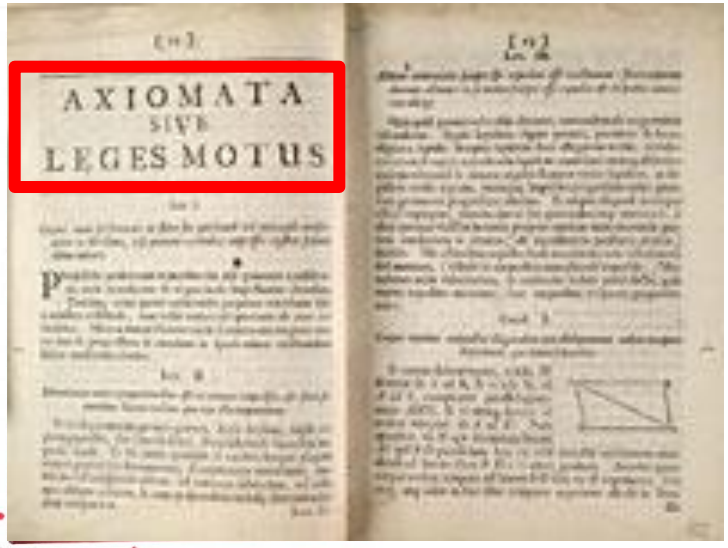
PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore *J. S. NEWTON*, *Trin. Coll. Cantab. Soc.* Matheſeos Profeſſore *Lucaſiano*, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,
Juſſu *Societatis Regiæ* ac Typis *Joſephi Streater*. Proſtat apud plures Bibliopolas. *Anno* MDCLXXXVII.

Isaac Newton,
Mathematical Principles of Natural Philosophy
(1687)


AXIOMATA
SIVE
LEGES MOTUS

AXIOMS, OR LAWS OF MOTION

LAW I

Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

Projectiles continue in their motions, so far as they are not retarded by the resistance of the air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are continually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by the air. The greater bodies of the planets and comets, meeting with less resistance in freer spaces, preserve their motions both progressive and circular for a much longer time.

LAW II

The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impressed altogether and at once, or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added to or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.

LAW III

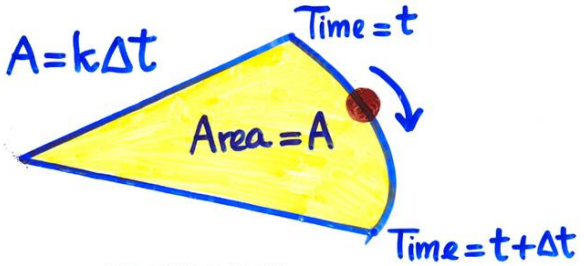
To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the

***Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy) by Isaac Newton (1687)**

Written in the language of Euclidean geometry

Isaac Newton, *Mathematical Principles of Natural Philosophy* (1687)



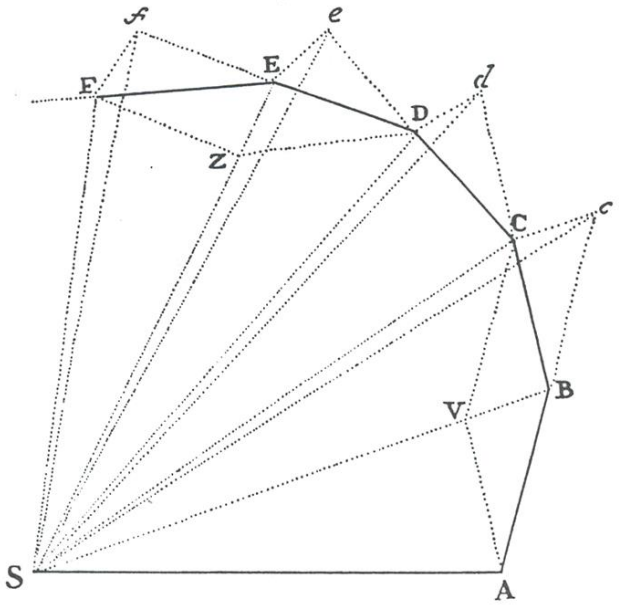
SECTION II

The determination of centripetal forces.

PROPOSITION I. THEOREM I

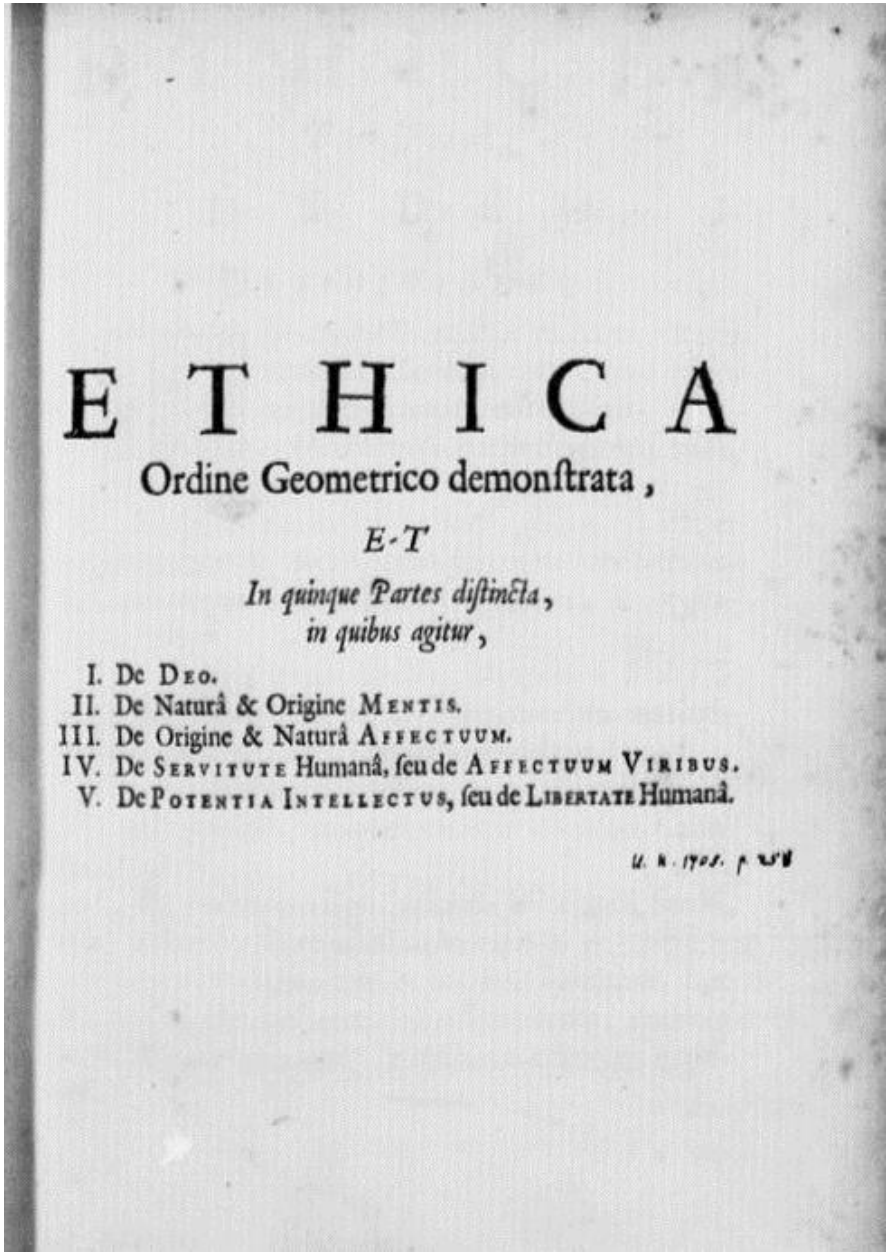
The areas which revolving bodies describe by radii drawn to an immovable centre of force do lie in the same immovable planes, and are proportional to the times in which they are described.

For suppose the time to be divided into equal parts, and in the first part of that time let the body by its innate force describe the right line AB. In the second part of that time, the same would (by Law 1), if not hindered,



proceed directly to *c*, along the line B*c* equal to AB; so that by the radii AS, BS, *c*S, drawn to the centre, the equal areas ASB, BS*c*, would be described. But when the body is arrived at B, suppose that a centripetal force acts at once with a great impulse, and, turning aside the body from the right line B*c*, compels it afterwards to continue its motion along the right line BC.

Spinoza's *Ethics* (1675)



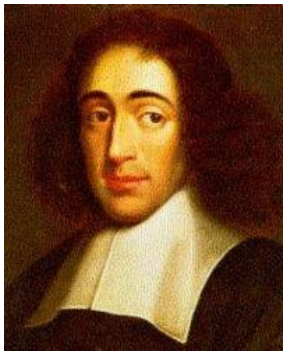
E T H I C A

Ordine Geometrico demonſtrata,

E-T

In quinque Partes diſtincta,
in quibus agitur,

I. De Deo.
II. De Naturâ & Origine Mentis.
III. De Origine & Naturâ Affectuum.
IV. De Servitute Humanâ, ſeu de Affectuum Viribus.
V. De Potentia Intellectus, ſeu de Libertate Humanâ.



Baruch Spinoza (1632-1677)

Baruch Spinoza, *Ethics* (1675)

SPINOZA'S ETHICS

ETHICS

PROVED IN GEOMETRICAL ORDER

(*Ethica ordine geometrico demonstrata*)

PART I

CONCERNING GOD

Baruch Spinoza, *Ethics* (1675)

essence (as I shall soon show). And therefore no reason can be given by which it can be said that God is passive to anything else than himself, or that extended substance is unworthy of divine nature, though it be supposed divisible, as long as it is granted to be eternal and infinite. But I have said enough of this at present.

PROP. XVI. Infinite things in infinite modes (that is, all things which can fall under the heading of infinite intellect) must necessarily follow from the necessity of divine nature.

Proof.—This proposition must be manifest to every one who will but consider this, that from a given definition of everything the intellect gathers certain properties, which in truth necessarily follow from the definition (that is, the very essence of the thing), and so the more reality the definition of a thing expresses, *i.e.*, the more reality the essence of a definite thing involves, the more properties the intellect will gather. But as divine nature has absolutely infinite attributes, each of which expresses infinite essence in its kind, infinite things in infinite modes (that is, all things that fall under the heading of infinite intellect) must necessarily follow its necessity. *Q.e.d.*

Corollary I.—Hence it follows that God is the effecting cause of all things which can be perceived by infinite intellect.

Corollary II.—Hence it follows that God is the cause through himself, and not indeed by accident.

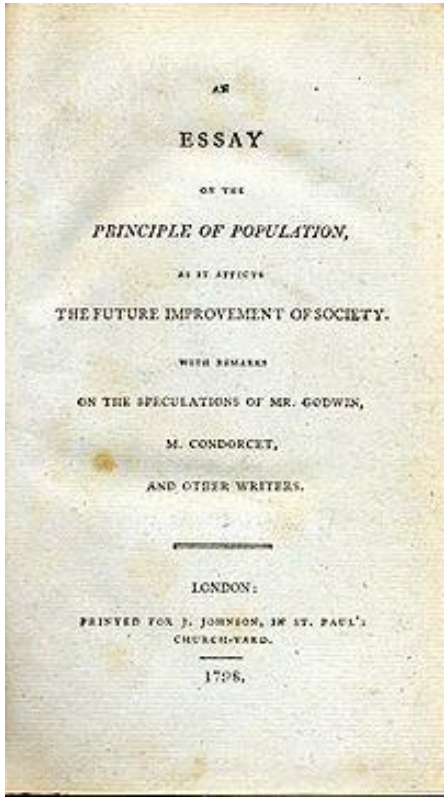
Corollary III.—Hence it follows that God is absolutely the first cause.

PROP. XVII. God acts merely according to his own laws, and is compelled by no one.

Proof.—That infinite things must follow from the mere necessity of divine nature, or what is the same thing, by the mere laws of divine nature, we have just shown (Prop. 16), and (Prop. 15) we have shown that nothing can be conceived without God, but that everything exists in God. Therefore nothing outside God can exist by which he could be determined or compelled in his actions; and therefore God acts merely according to the laws of his nature, and is compelled by no one. *Q.e.d.*

Corollary I.—Hence it follows that no cause can be given except the perfection of God's nature which extrinsically or intrinsically incites him to action.

Q.e.d
= Quod Erat Demonstratum



AN

ESSAY

ON THE

PRINCIPLE OF POPULATION,

AS IT AFFECTS

THE FUTURE IMPROVEMENT OF SOCIETY.

WITH REMARKS

ON THE SPECULATIONS OF MR. GODWIN,

M. CONDORCET,

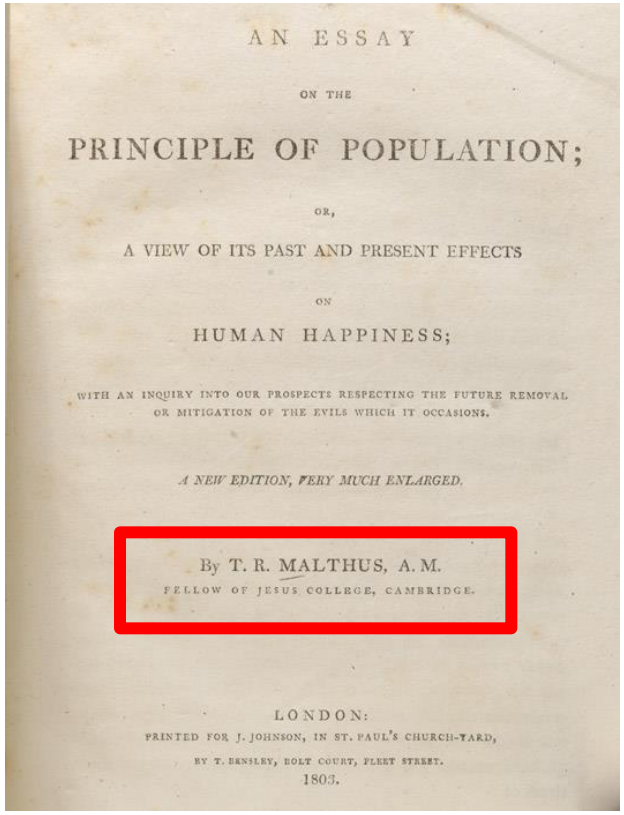
AND OTHER WRITERS.

LONDON:

PRINTED FOR J. JOHNSON, IN ST. PAUL'S CHURCH-YARD.

1798.

***An Essay on the Principle of Population* by Thomas Malthus (published anonymously in 1798)**



AN ESSAY

ON THE

PRINCIPLE OF POPULATION;

OR,

A VIEW OF ITS PAST AND PRESENT EFFECTS

ON

HUMAN HAPPINESS;

WITH AN INQUIRY INTO OUR PROSPECTS RESPECTING THE FUTURE REMOVAL OR MITIGATION OF THE EVILS WHICH IT OCCASIONS.

A NEW EDITION, VERY MUCH ENLARGED.

By T. R. MALTHUS, A.M.

FELLOW OF JESUS COLLEGE, CAMBRIDGE.

LONDON:

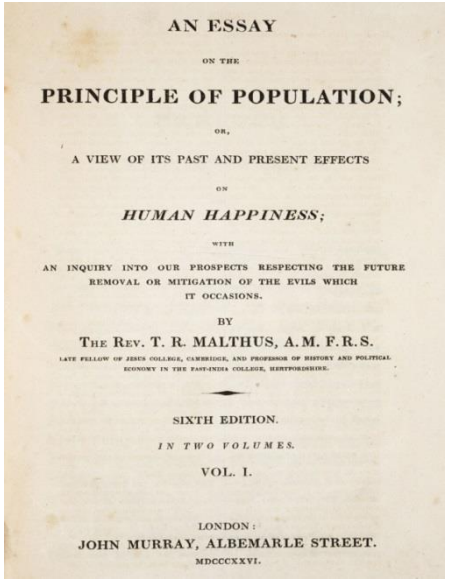
PRINTED FOR J. JOHNSON, IN ST. PAUL'S CHURCH-YARD,

BY T. BENSLEY, BOLT COURT, FLEET STREET.

1803.

Thomas Robert Malthus (1766-1834)

Many subsequent editions, with the final 6th edition published in 1826



AN ESSAY

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BY

THE REV. T. R. MALTHUS, A.M. F.R.S.

LATE FELLOW OF JESUS COLLEGE, CAMBRIDGE, AND PROFESSOR OF HISTORY AND POLITICAL ECONOMY IN THE EAST-INDIA COLLEGE, HERTFORDSHIRE.

SIXTH EDITION.

IN TWO VOLUMES.

VOL. I.

LONDON:

JOHN MURRAY, ALBEMARLE STREET.

MDCCCXXVI.

"It has been said that the great question is now at issue, whether man shall henceforth start forwards with accelerated velocity towards illimitable, and hitherto unconceived improvement; or be condemned to a perpetual oscillation between happiness and misery, and after every effort remain still at an immeasurable distance from the wished-for goal.

. . . . . .

I think I may fairly make two postulata.

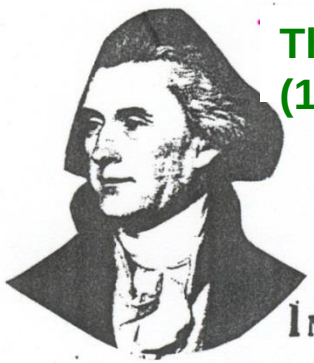
First, that food is necessary to the existence of man.

Secondly, That the passion between the sexes is necessary, and will remain nearly in its present state.

. . . . . .

Assuming, then, my postulata as granted, I say, that the power of population is indefinitely greater than the power in the earth to produce subsistence for man."

Thomas Malthus
"An Essay on the Principle of Population", Chapter I
(first edition, 1798)



Thomas Jefferson
(1743-1826)

IN CONGRESS. JULY 4. 1776

The unanimous Declaration of the thirteen united States of America.

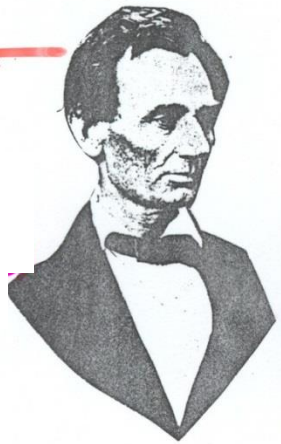
• • • • • We hold these truths to be self-evident, that all men are created equal, that they are endowed by their Creator with certain unalienable Rights, that among these are Life, Liberty and the pursuit of Happiness. That to secure these rights, Governments are instituted among Men, deriving their just powers from the consent of the governed. That whenever any Form of Government becomes destructive of these ends, it is the Right of the People to alter or to abolish it, and to institute new Government, laying its foundation on such principles and organizing its powers in such form, as to them shall seem most likely to effect their Safety and Happiness. • • • • •

Declaration of Independence 1776

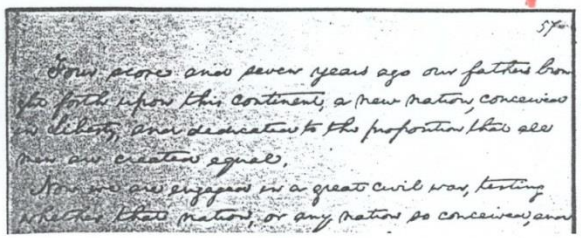
"One would start with great confidence that he could convince any sane child that the simpler propositions of Euclid are true; but nevertheless, he would fail, utterly, with one who should deny the definitions and axioms. The principles of Jefferson are the definitions and axioms of free society."

A. Lincoln to H.L. Pierce, 6 Apr. 1859

Four score and seven years ago our fathers brought forth on this continent, a new nation, conceived in Liberty, and dedicated to the proposition that all men are created equal. ···



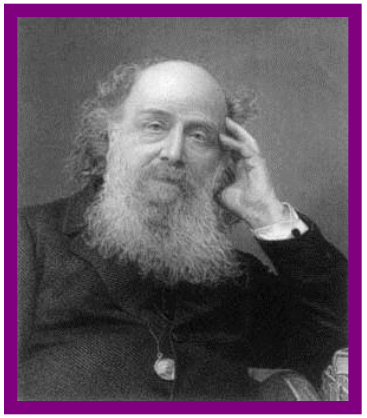
Abraham Lincoln
Address delivered at the
dedication of the cemetry
at Gettysburg, 19 Nov. 1863



Abraham Lincoln (1809-1865)

"The early study of Euclid made me a hater of Geometry, which I hope may plead my excuse if I have shocked the opinions of any in this room (...) by the tone in which I have previously alluded to it as a schoolbook; "

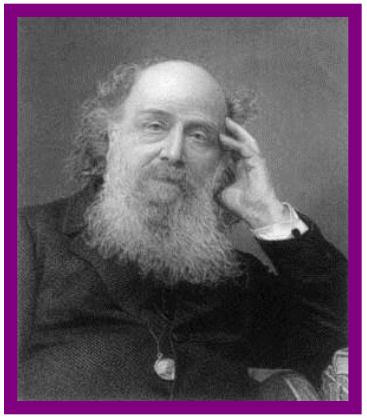
The Collected Mathematical Papers of James Joseph Sylvester, Volume II, edited by H.F. Baker (four volumes, 1904-1910).



James Joseph Sylvester (1814-1897)

"and yet, in spite of this repugnance, which had become a second nature in me, whenever I went far enough into any mathematical question, I found I touched, at last, a geometrical bottom."

The Collected Mathematical Papers of James Joseph Sylvester, Volume II, edited by H.F. Baker (four volumes, 1904-1910).



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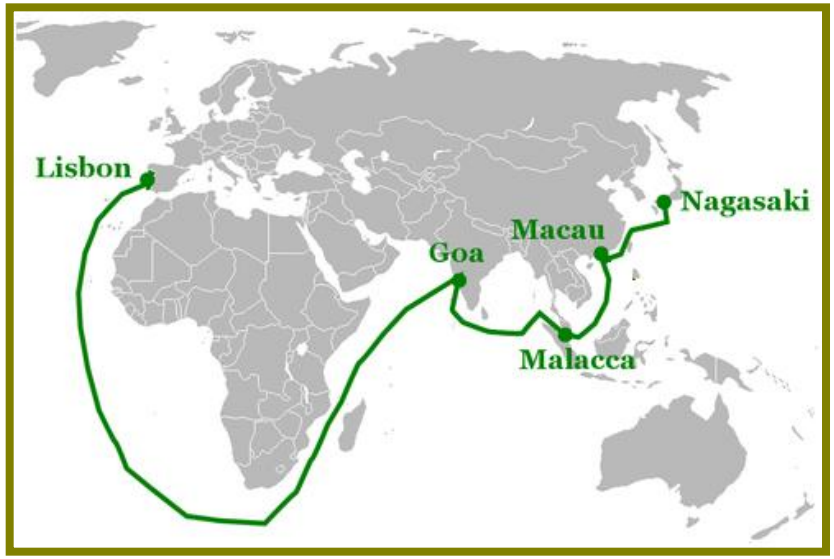
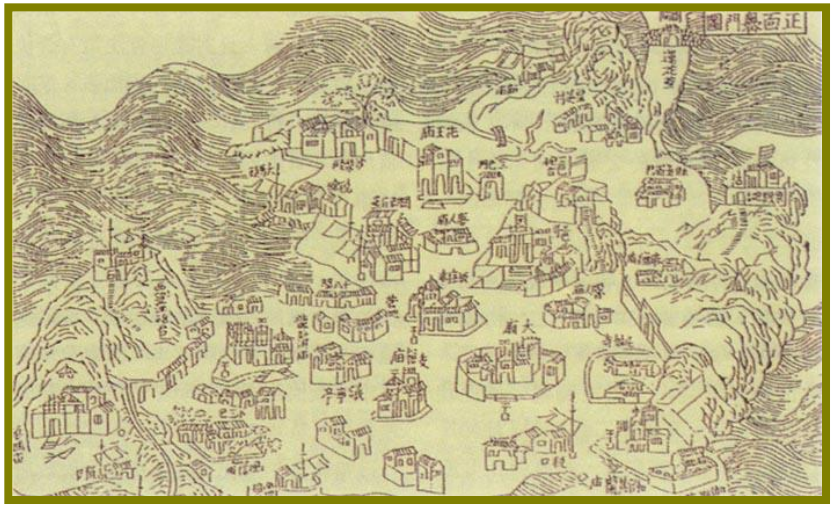
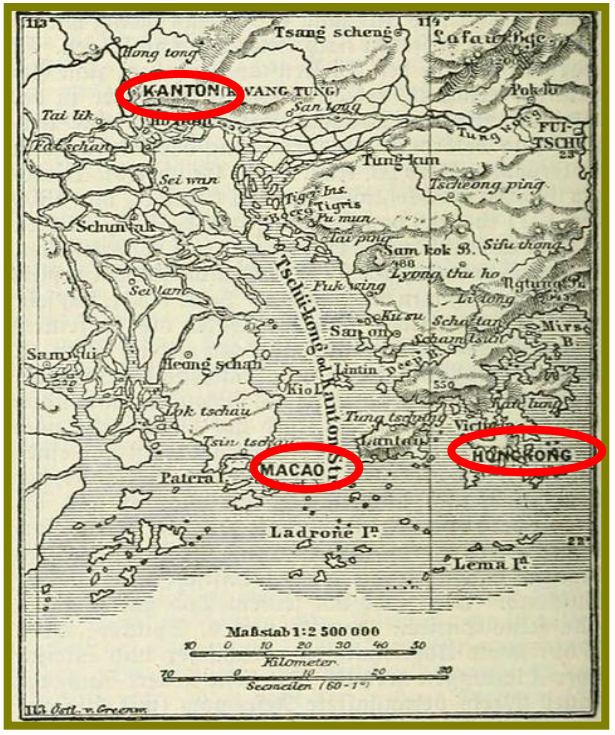


The Portuguese established posts at Goa in 1510 and at Malacca in 1511.
In 1557 the Ming Court gave consent for establishment of an official Portuguese trade post at Macau.



Town map of Macau
(17th / 18th century ?)



Map of Macau, Hong Kong and Canton [Guangzhou] (late 19th century)

A statue of Matteo Ricci was erected on August 7, 2010 at the archeological remains of Colégio de Säo Paulo (St. Paul's College) in Macau.

St. Paul's College, founded by the Jesuit Alessandro Valignano (1539-1606) in 1594, was the first western–style university in the Far East.

Matteo Ricci
利瑪竇
(1552-1610)

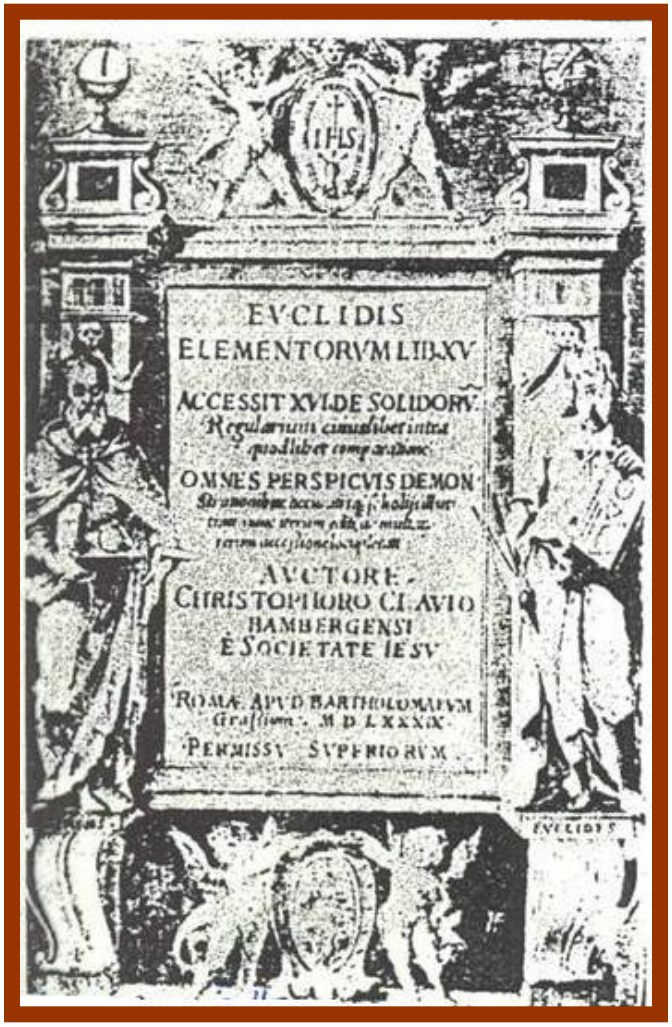
Copper engraving of the XVIIth Century. (*The Bettmann Archive*)

XU Guang-qi
徐光啟
(1562-1633)

Tomb of Matteo Ricci (1552-1610) in Beijing

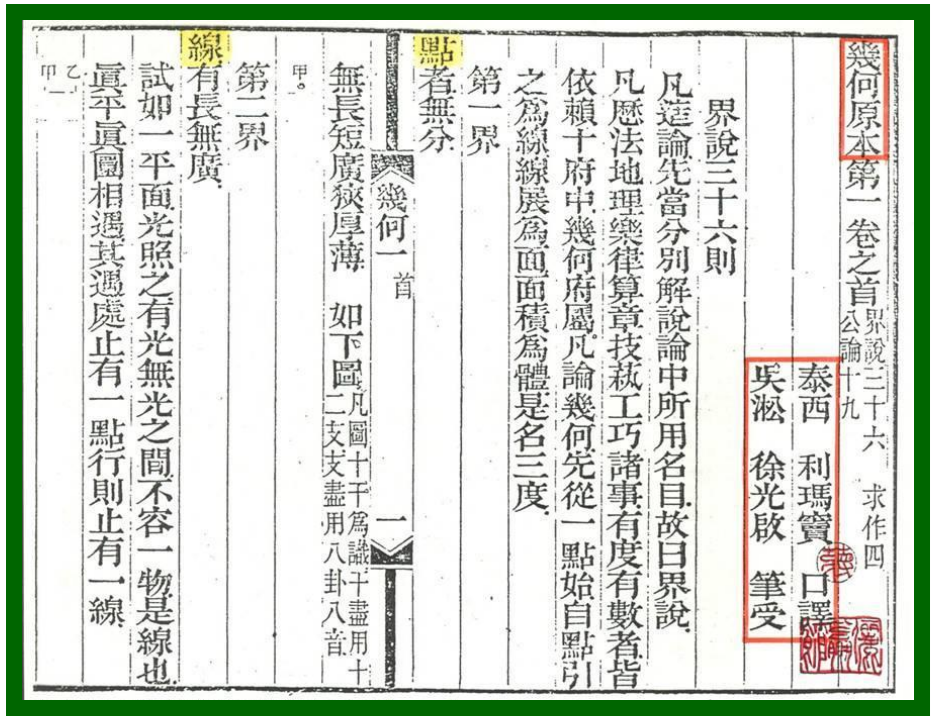
利瑪竇 (Matteo Ricci)
1552 - 1610

徐光啟 (XU Guang-qi)
1562 - 1633



C. CLAVIUS,
EUCLIDIS ELEMENTORUM LIBRI XV
(1574; 1589)

EUCLID'S
ELEMENTS
(c. 300 B.C.E.)


幾何原本第一卷之首界說三十六 求作四
公論十九
泰西 利瑪竇 口譯
吳淞 徐光啟 筆受
界說三十六則
凡造論先當分別解說論中所用名目故曰界說
凡歷法地理樂律算章技藝工巧諸事有度有數者皆
依賴于府中幾何府屬凡論幾何先從一點始自點引
之爲線線展爲面面積爲體是名三度
第一界
點者無分
幾何一首
一
甲。
無長短廣狹厚薄 如下圖凡圖十干爲識干盡用十
二支支盡用八卦八音
第二界
線有長無廣
試如一平面光照之有光無光之間不容一物是線也
直平直圖相遇其遇處止有一點行則止有一線
乙
甲

幾何原本

利瑪竇與徐光啟
合譯的中文本 (1607)

"… Whoever may think that ethics, physics and mathematics are not important in the work of the Church, is unacquainted with the taste of the Chinese, who are slow to take a salutary spiritual potion, unless it be seasoned with an intellectual flavouring. […]

All this, what we have recounted relative to a knowledge of science, served as seed for a future harvest, and also as a foundation for the nascent Church in China…"

China in the Sixteenth Century: The Journals of Matthew Ricci, 1583-1610 **[compiled by Nicolas Trigault and published in 1615; translated from Latin into English by L.J. Gallagher in 1942; 1953]**

"… but nothing pleased the Chinese as much as the volume on the Elements of Euclid. This perhaps was due to the fact that no people esteem mathematics as highly as the Chinese, despite their method of teaching, in which they propose all kinds of propositions but without demonstrations."

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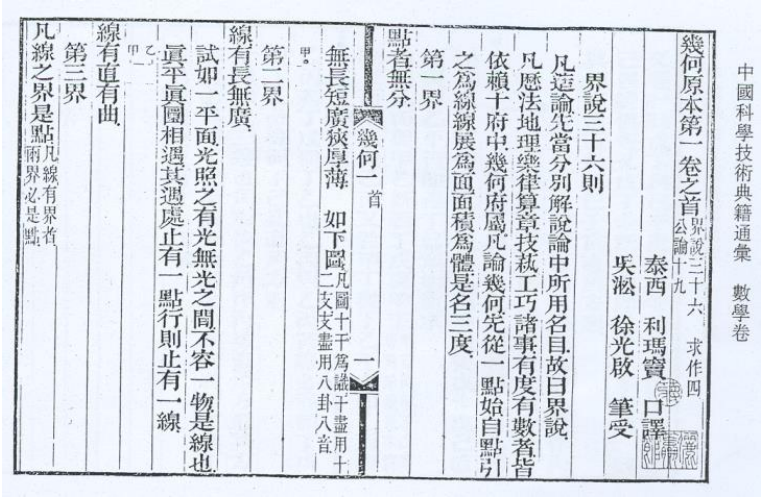
當真?

「太史意方銳，欲竟之。余曰：止，請先傳此，使同志習之，果以為用也，而後徐計其餘。太史曰：然，是書也，苟為用，竟之何必在我。遂輟譯而梓。」

利瑪竇
《幾何原本》序 (1607)

「續成大業，未知何日，未知何人。」
徐光啟•《幾何原本》修訂版序 (1611)


中國科學技術典籍通彙　數學卷

幾何原本第一卷之首　界說二十六　求作四　公論十九

泰西　利瑪竇　口譯

吳淞　徐光啟　筆受

界說三十六則

凡這論先當分別解說論中所用名目故曰界說

凡歷法地理樂律算章技藝工巧諸事有度有數者皆依賴于府中幾何府屬凡論幾何先從一點始自點引之爲線線展爲面面積爲體是名三度

第一界

點者無分

幾何一首

無長短廣狹厚薄　如下圖凡圖十干爲識十盡用十支支盡用八卦八音

甲

第二界

線有長無廣

試如一平面光照之有光無光之間不容一物是線也

甲　乙

眞平眞圓相遇其遇處止有一點行則止有一線

線有直有曲

第三界

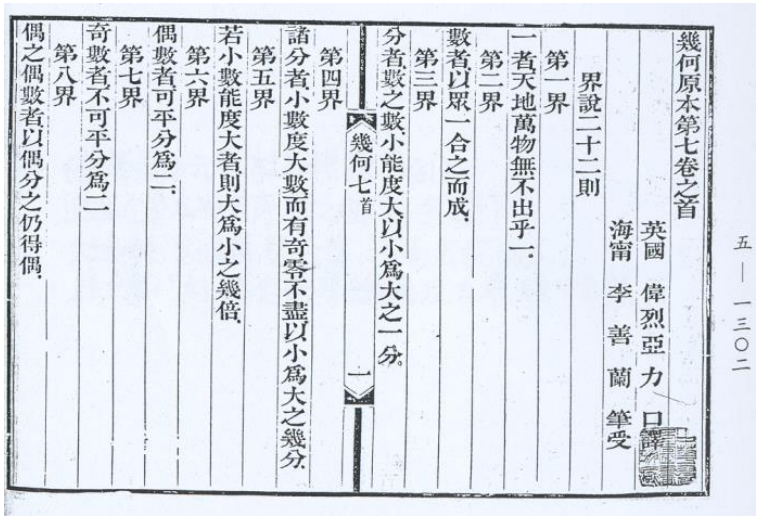
凡線之界是點凡線有界者兩界必是點

translation by XU Guangqi and Matteo Ricci (1607)

Book I to *Book VI*

(based on Latin compilation by Christopher Clavius, 1574/1589)

250 years !


幾何原本第七卷之首

英國　偉烈亞力　口譯

海甯　李善蘭　筆受

界說二十二則

第一界

一者天地萬物無不出乎一

第二界

數者以衆一合之而成

第三界

分者數之數小能度大以小爲大之一分

幾何七首

第四界

諸分者小數度大數而有奇零不盡以小爲大之幾分

第五界

若小數能度大者則大爲小之幾倍

第六界

偶數者可平分爲二

第七界

奇數者不可平分爲二

第八界

偶之偶數者以偶分之仍得偶

五—一三〇二

translation by LI Shanlan and Alexander Wylie (1857)

Book VII to *Book XV*

(based on English translation by Henry Billingsley, 1570)

Courtesy from the Hong Kong University Libraries

《幾何原本十五卷》金陵足本 (1857/1865) [偉烈亞力(Alexander Wylie)口譯，李善蘭筆授，於1857年刊行，惜不久即遇上太平兵變及英法聯軍入侵，版燬無傳。遞至曾國藩駐守金陵(即今南京)，李善蘭向曾氏述及此書之重要，曾氏逐出資重印該書，十五卷(前六卷乃明代徐光啟與利瑪竇(Matteo Ricci) 合譯之刻本)於1895年再現中土。]

Translation of *Book VII* to *Book XV* of *Elements* by LI Shanlan and Alexander Wylie (1857), completing the translation of (fifteen books of) *Elements*.

徐光啟如何認識他剛從Clavius編纂的Euclid 的*Elements* 學到的幾何呢？

他又如何理解書中的思想、方法和表達形式，那些都與他熟悉的中國傳統數學很不相同？

蕭文強, 當「歐先生」來到中國 … ,
《數學傳播》, 38(4)(2014), 24-41.

雖然徐光啟強調數學的應用，
他有足夠的視野洞識《原本》
本質的特點。

在《幾何原本》刻本序言(1607)
他寫道：

「由顯入微，從疑得信，
蓋不用為用，眾用所基，
真可謂萬象之形囿，百家
之學海。」

「西泰子之譯測量諸法也，十年矣。法而系之義也，自歲丁未始也。曷待乎？於時幾何原本之六卷始卒業矣，至是而後能傳其義也。是法也、與周髀九章之句股測望、異乎？不異也。不異，何貴焉？亦貴其義也。」

徐光啟・《題測量法義》1608

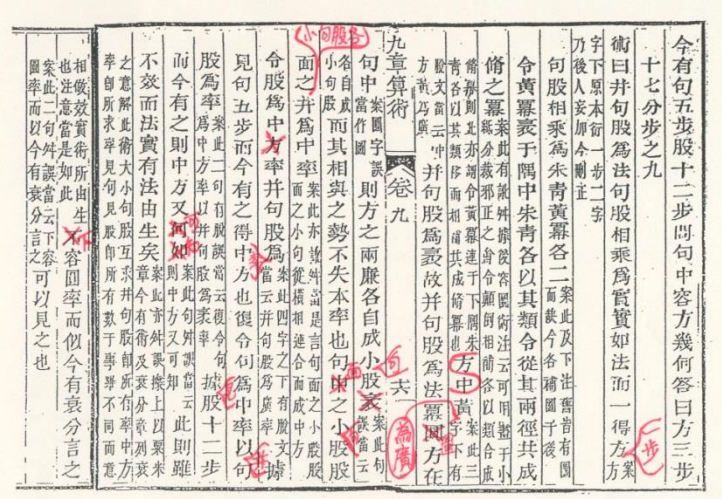
「其義全闕，學者不能識其所繇。既具新論，以考舊文，如視掌矣。」

徐光啟・《測量異同緒言》1608

「不知其中有理、有義、有法、有數。理不明不能立法，義不辨不能著數。明理辨義，推究頗難；法立數著，遵循甚易。」

徐光啟・《測候月食奉旨回奏疏》1629

Problem 15 in Chapter 9 of *Jiuzhang Suanshu* (九章算術)



今有句五步股十二步問句中容方幾何答曰方三步

十七分步之九

術曰并句股爲法句股相乘爲實實如法而一得方步

句股相乘爲朱青黃冪各二

令黃冪衰于隅中朱青各以其類令從其兩徑共成

脩之冪

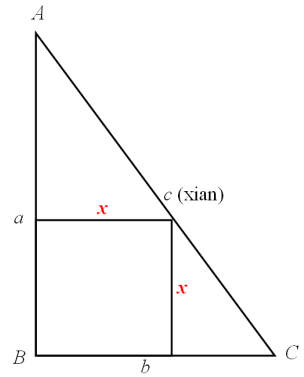
不衰而法實有法由生矣

令股爲中方率并句股爲廣率并句股爲長率

見句五步而今有之得中方也復令句爲中率以句

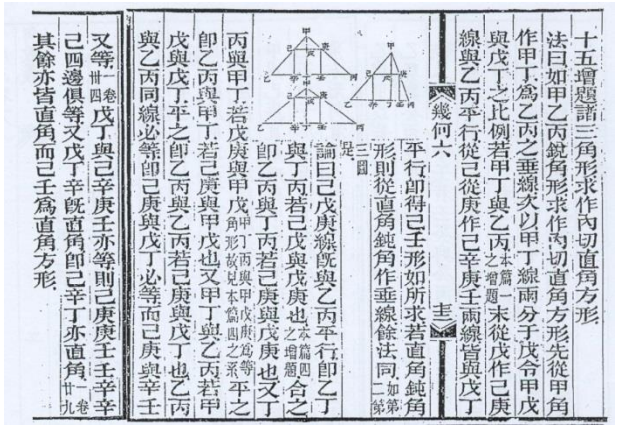
股爲率

而今有之則中方又可知



$$x = \frac{ab}{a+b}$$

Added Proposition 15 of Book VI in *Euclidis Elementorum Libri XV*



十五增題諸三角形求作內切直角方形

法曰如甲乙丙銳角形求作內切直角方形先從甲角

作甲丁爲乙丙之垂線次以甲丁線兩分于戊令甲戊

與戊丁之比例若甲丁與乙丙本篇一末之增題從戊作己庚

線與乙丙平行從己從庚作己辛庚壬兩線皆與戊丁

幾何六

平行即得己壬形如所求若直角鈍角

形則從直角鈍角作垂線餘法同加第

論曰己戊庚線既與乙丙平行即乙丁

與丁丙若己戊與戊庚也本篇四之增題合之

即乙丙與丁丙若己庚與戊庚也又丁

丙與甲丁若戊庚與甲戊角甲丁丙與甲戊形故見本篇四之末平之

即乙丙與甲丁若己庚與甲戊也又甲丁與乙丙若甲

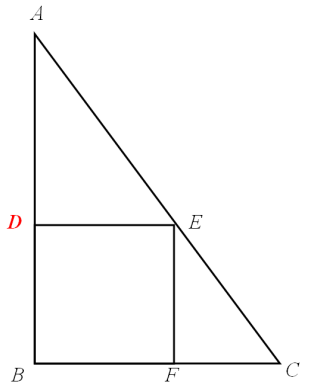
戊與戊丁平之即乙丙與乙丙若己庚與戊丁也乙丙

與乙丙同線必等即己庚與戊丁必等而己庚與辛壬

又等一卷廿四戊丁與己辛庚壬亦等則己庚壬壬辛辛

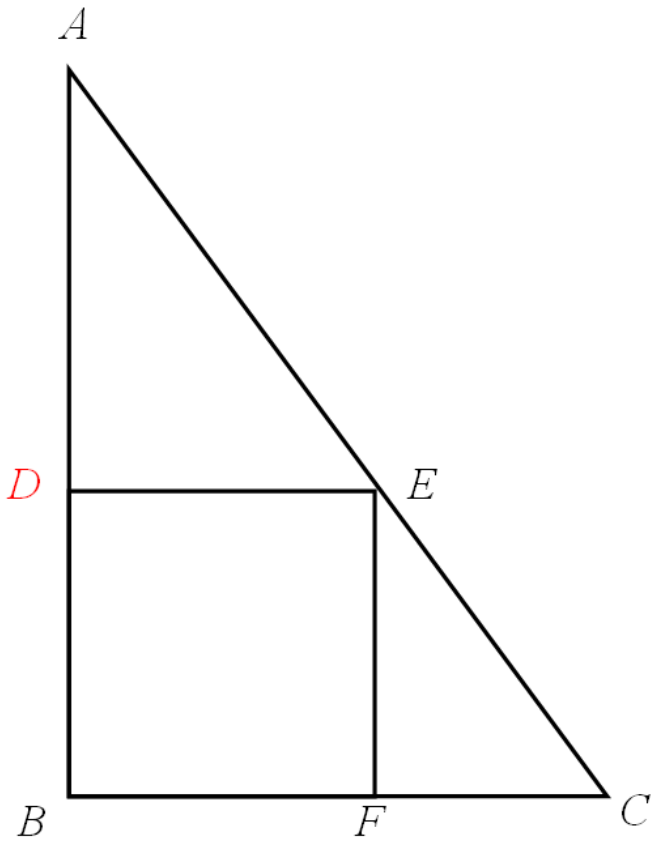
己四邊俱等又戊丁辛既直角即己辛丁亦直角一卷廿九

其餘亦皆直角而己壬爲直角方形

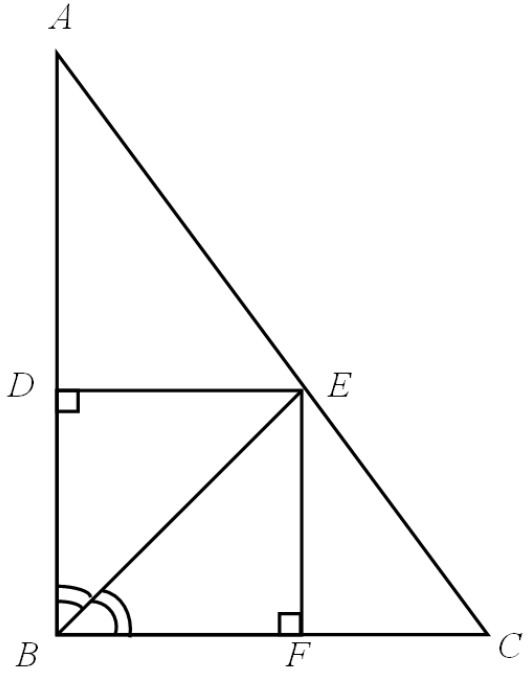


$$AD : DB = AB : BC$$

Given a right-angled triangle *ABC* with *AC* as its hypotenuse, inscribe a square in it, that is, construct a square *BDEF* with *D* on *AB*, *E* on *AC*, and *F* on *BC*?



This problem does not appear in Euclid's *Elements*. Were it there, the solution would have probably looked like this.



Bisect $\angle ABC$ by BE (E on AC)

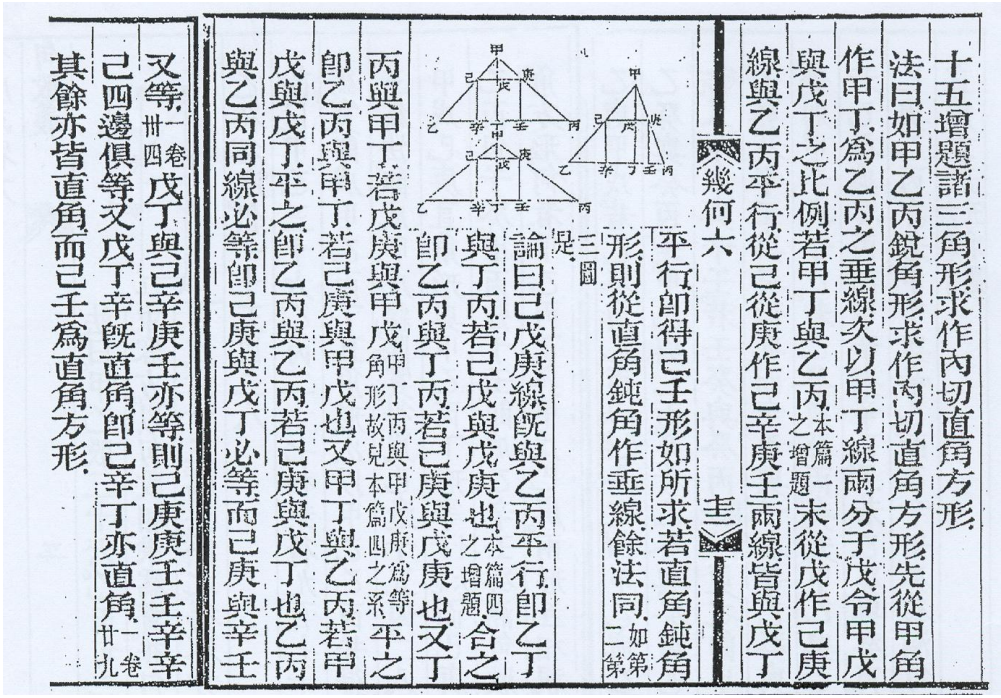
[Book I, Proposition 9]

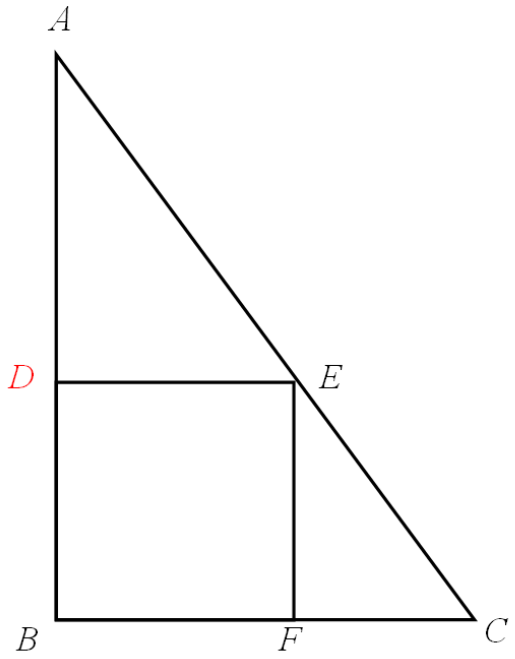
Drop perpendiculars ED, EF (D on AB, F on BC) [Book I, Proposition 12]

Prove that $BDEF$ is the inscribed square we want.

Exercise for the (modern) reader: Show that the side of the inscribed square is equal to $\frac{ab}{a+b}$, where $AB = a$ and $BC = b$.

But the problem (in a more general version) appears as Added Proposition 15 of Book VI in *Euclidis Elementorum Libri XV*, which was translated by Matteo Ricci and XU Guangqi.


十五增題諸三角形求作內切直角方形
法曰如甲乙丙銳角形求作內切直角方形先從甲角作甲丁為乙丙之垂線次以甲丁線兩分于戊令甲戊與戊丁之比例若甲丁與乙丙本篇一之增題末從戊作己庚線與乙丙平行從己從庚作己辛庚壬兩線皆與戊丁
幾何六 圭
平行即得己壬形如所求若直角鈍角形則從直角鈍角作垂線餘法同如第二第三圖
是
論曰己戊庚線既與乙丙平行即乙丁與丁丙若己戊與戊庚也本篇四之增題合之即乙丙與丁丙若己庚與戊庚也又丁丙與甲丁若戊庚與甲戊甲丁丙與甲戊庚為等角形故見本篇四之系平之即乙丙與甲丁若己庚與甲戊也又甲丁與乙丙若甲戊與戊丁平之即乙丙與乙丙若己庚與戊丁也乙丙與乙丙同線必等即己庚與戊丁必等而己庚與辛壬又等一卷廿四戊丁與己辛庚壬亦等則己庚庚壬壬辛辛己四邊俱等又戊丁辛既直角即己辛丁亦直角一卷廿九其餘亦皆直角而己壬為直角方形



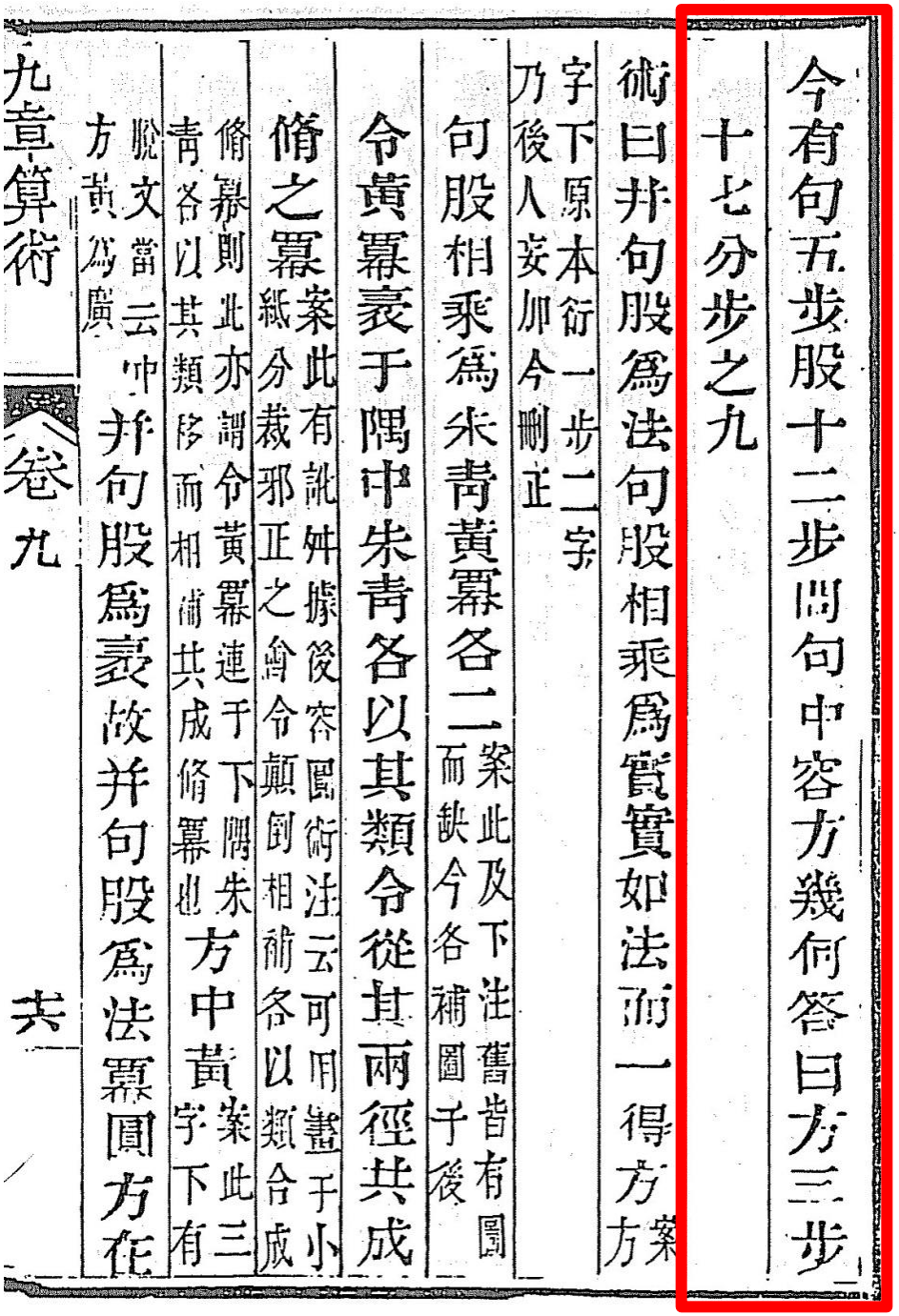
The solution is like this.

Divide *AB* at *D* such that

AD* : *DB* = *AB* : *BC

[Book VI, Proposition 10]

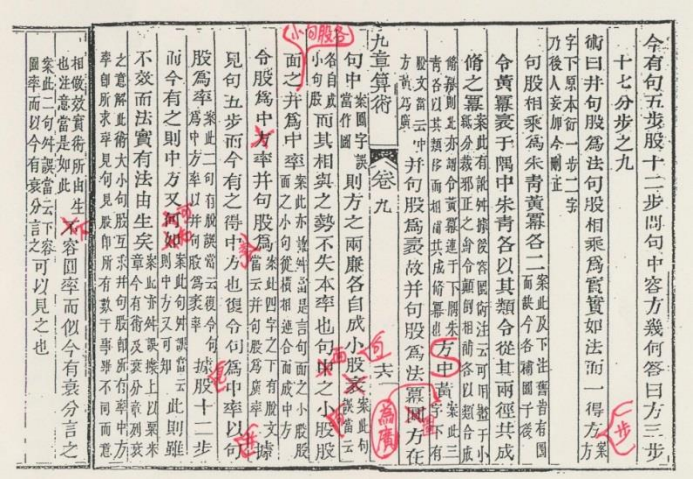
Draw *DE* parallel to *BC* and *EF* parallel to *AB*, (*E* on *AC*, *F* on *BC*).
***DBFE* is the inscribed square we want.**


今有句五步股十二步問句中容方幾何答曰方三步
十七分步之九
術曰并句股爲法句股相乘爲實實如法而一得方案
字下原本衍一步二字
乃後人妄加今刪正
方
句股相乘爲朱青黃冪各二案此及下注舊皆有圖
而缺今各補圖于後
令黃冪袤于隅中朱青各以其類令從其兩徑共成
脩之冪案此有訛舛據後容圓術注云可用畫于小
紙分裁邪正之會令顛倒相補各以類合成
脩冪則此亦謂令黃冪連于下隅朱
青各以其類陪而相補共成脩冪也方中黃案此三
脫文當云中
字下有
方黃爲廣并句股爲袤故并句股爲法冪圓方在
九章算術 卷九 十六

《九章算術》
第九章第十五題

今有句五步，
股十二步。
問：句中容方幾何？
答曰：方三步一十七分步之九。

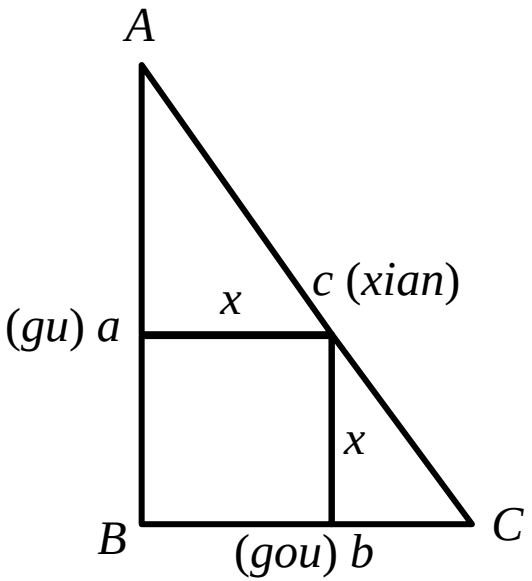
《九章算術》· 第九章 · 第十五題



今有句五步，股十二步。問：句中容方幾何？答曰：方三步一十七分 步之九。

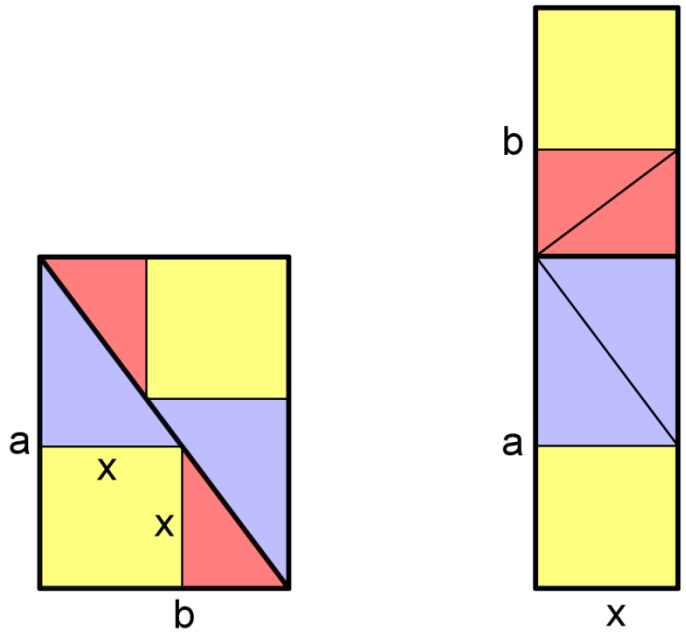
術曰：并句、股為法，句、股相乘為實。實如法而一，得方一步。

$$x = \frac{ab}{a+b}$$



劉徽注《九章算術》（公元三也世紀中葉）

方法 1 (出入相補)



Area = ab Area = $(a + b)\,x$

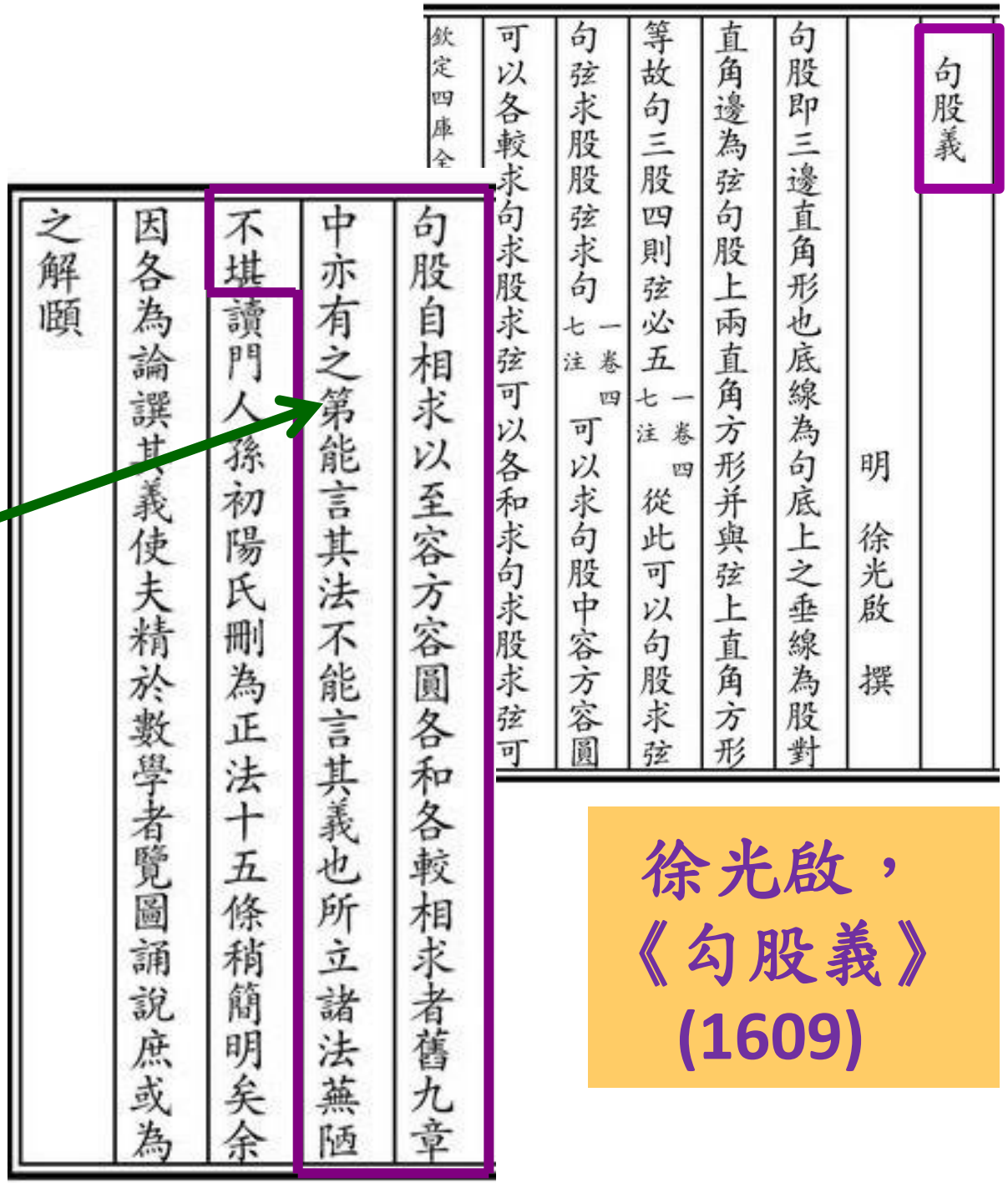
$$ab = (a + b)\,x$$

$$x = \frac{ab}{a + b}$$

http://ggbtu.be/m2812253

方法 2 (率與比例)

— 省略掉！



句股義

明　徐光啟　撰

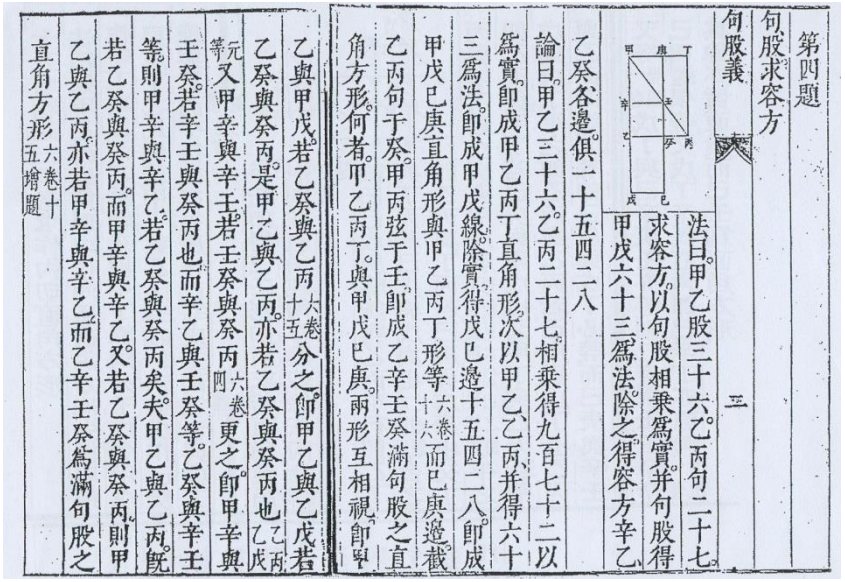
句股即三邊直角形也底線為句底上之垂線為股對直角邊為弦句股上兩直角方形并與弦上直角方形等故句三股四則弦必五（一卷四七注）從此可以句股求弦句弦求股股弦求句（一卷四七注）可以求句股中容方容圓可以各較求句求股求弦可以各和求句求股求弦可

欽定四庫全

句股自相求以至容方容圓各和各較相求者舊九章中亦有之第能言其法不能言其義也所立諸法蕪陋不堪讀門人孫初陽氏刪為正法十五條稍簡明矣余因各為論譔其義使夫精於數學者覽圖誦說庶或為之解頤

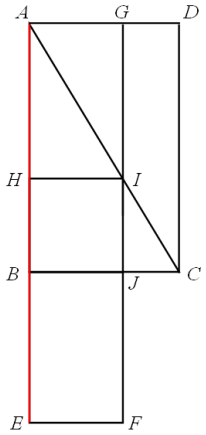
第能言其法
不能言其義也

徐光啟，
《勾股義》
(1609)

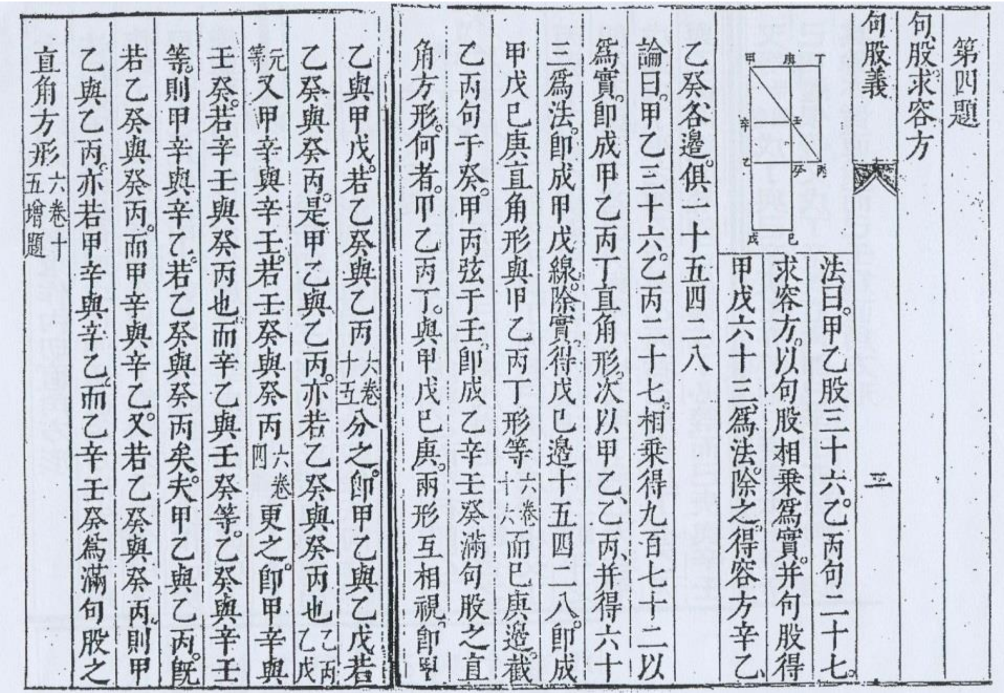

第四題
句股求容方
句股義 三
法曰甲乙股三十六乙丙句二十七求容方以句股相乘爲實并句股得甲戊六十三爲法除之得容方辛乙乙癸各邊俱一十五四二八
論曰甲乙三十六乙丙二十七相乘得九百七十二以爲實即成甲乙丙丁直角形次以甲乙乙丙并得六十三爲法即成甲戊線除實得戊巳邊十五四二八即成甲戊巳庚直角形與甲乙丙丁形等乙丙句于癸甲丙弦于壬即成乙辛壬癸滿句股之直角方形何者甲乙丙丁與甲戊巳庚兩形互相視即皆
乙與甲戊若乙癸與乙丙分之即甲乙與乙戊若乙癸與癸丙是甲乙與乙丙亦若乙癸與癸丙也又甲辛與辛壬若壬癸與癸丙更之即甲辛與壬癸若辛壬與癸丙也而辛乙與壬癸等乙癸與辛壬等則甲辛與辛乙若乙癸與癸丙矣夫甲乙與乙丙既乙與乙丙亦若甲辛與辛乙又若乙癸與癸丙則甲辛與辛乙而乙辛壬癸爲滿句股之直角方形何者甲乙丙丁與甲戊巳庚兩形互相視即皆

徐光啟
《勾股義》
1609

第四題： 句股求容方。法曰:甲乙股三十六，乙丙句二十七，求容方。以句股相乘為實，并句股得甲戊六十三為法，除之得容方辛乙、乙癸、各邊。俱一十五四二八。…



已知容方邊長為句乘股除以句加股。先製作 *AEFG* 與 *ABCD* 等面積，*AEFG* 的一邊是句加股 (*BE* = *BC*)，另一邊便是求作容方的邊。由此證明點 *H* 分割 *AB* 滿足 *AH:HB = AB:BC*。符合附加命題十五的要求。



第四題

句股求容方

句股義　三

法曰。甲乙股三十六。乙丙句二十七。求容方。以句股相乘為實。并句股得甲戊六十三為法。除之。得容方辛乙。乙癸各邊。俱一十五四二八

論曰。甲乙三十六。乙丙二十七。相乘得九百七十二。以為實。即成甲乙丙丁直角形。次以甲乙。乙丙。并得六十三為法。即成甲戊線。除實。得戊己邊十五四二八。即成甲戊己庚直角形。與甲乙丙丁形等。六卷十六而己庚邊截乙丙句于癸。甲丙弦于壬。即成乙辛壬癸滿句股之直角方形。何者。甲乙丙丁與甲戊己庚。兩形互相視。即甲乙與甲戊。若乙癸與乙丙。六卷十五分之。即甲乙與乙戊。若乙癸與癸丙。是甲乙與乙丙。亦若乙癸與癸丙也。又甲辛與辛壬。若壬癸與癸丙。六卷四更之。即甲辛與壬癸。若辛壬與癸丙也。而辛乙與壬癸等。乙癸與辛壬等。則甲辛與辛乙。若乙癸與癸丙矣。夫甲乙與乙丙。既若乙癸與癸丙。而甲辛與辛乙。又若乙癸與癸丙。則甲乙與乙丙。亦若甲辛與辛乙。而乙辛壬癸為滿句股之直角方形。六卷十五增題

徐光啟
《勾股義》
1609

徐光啟在書中用到的複雜推論，

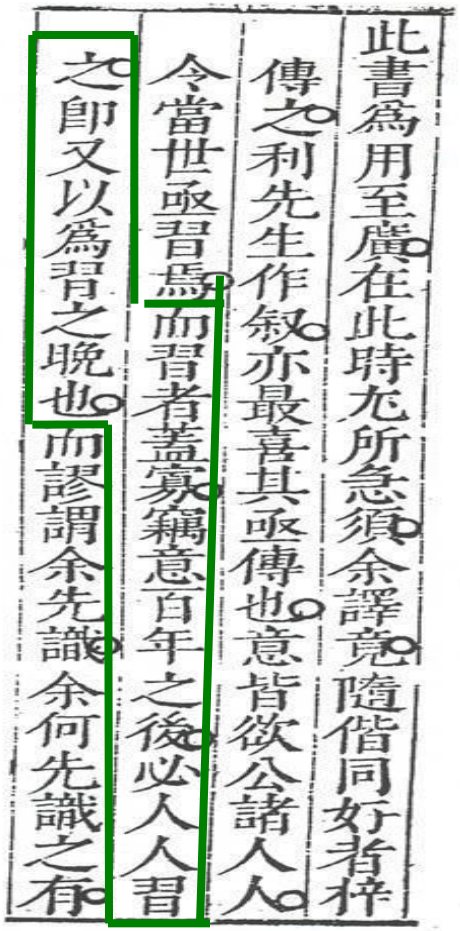
看來迂迴而且非必要。

可能，這顯示了西方與中國處理數學的方式有某種不協調，

勉強把一種方式塑造成另一種，

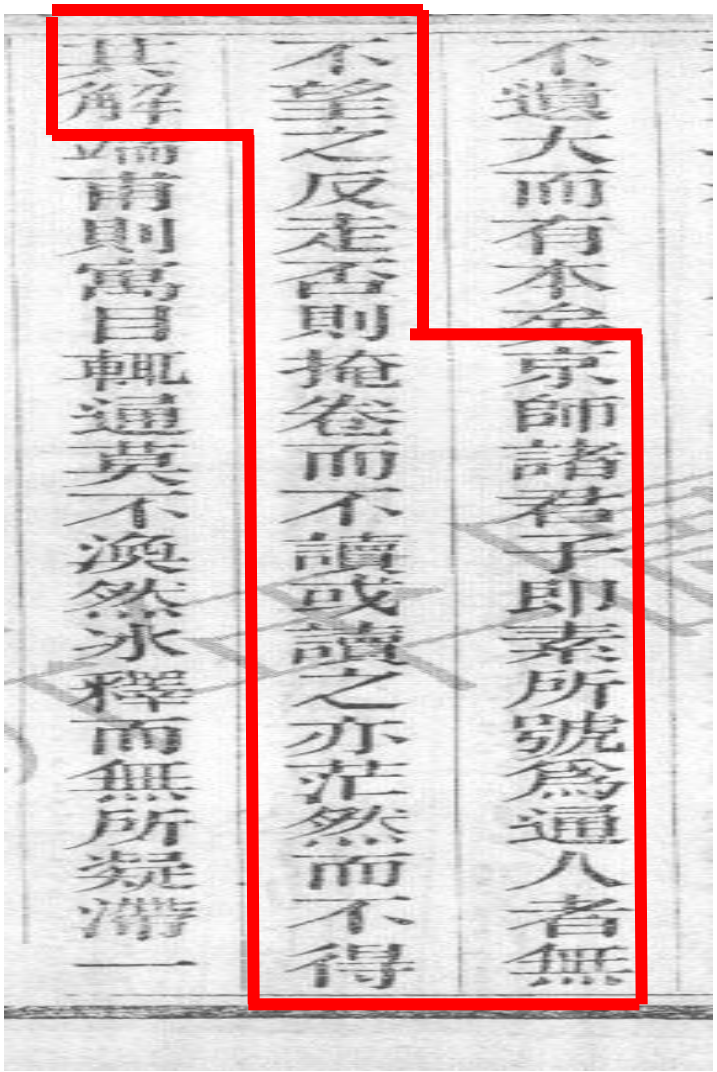
硬套進去，便顯得很不自然了。

"This book [the *Elements*] has wide applications and is particularly needed at this point in time. …… In the preface Mister Ricci also expressed his wish to promulgate this book so that it can be made known to everybody who will then study it. **Few people study it. I surmise everybody will study it a hundred years from now, at which time they will regret that they study it too late.** They would wrongly attribute to me the foresight [in introducing this book], but what foresight have I really?"


此書爲用至廣在此時尤所急須余譯竟隨偕同好者梓
傳之利先生作叙亦最喜其亟傳也意皆欲公諸人人
令當世亟習焉而習者蓋寡竊意百年之後必人人習
之即又以爲習之晚也而謬謂余先識余何先識之有

徐光啟·幾何原本雜議

XU Guang-qi, *Various Reflections on Jihe Yuanben* (徐光啟，《幾何原本雜議》), 1607

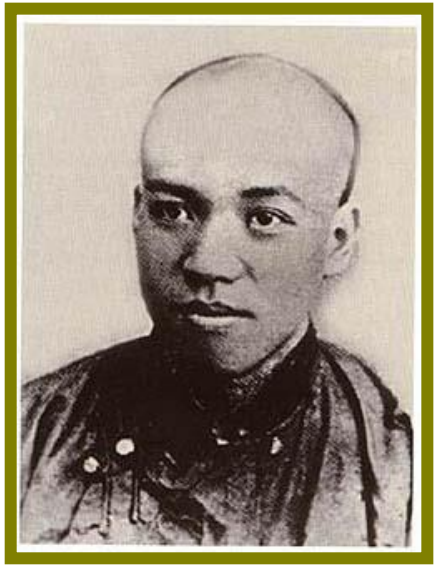


“Even those gentlemen in the capital who regard themselves to be erudite scholars keep away from the book, or close it and do not study its content at all, or study it with incomprehension and perplexity.”

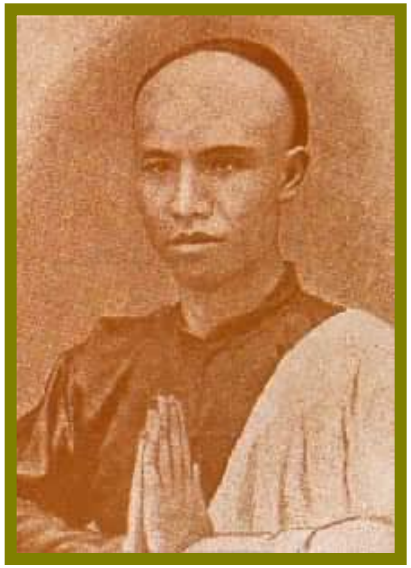
Li Zi-jin [李子金], Preface to
The Key to Mathematics
(Du Zhi-geng [杜知耕] , 1681)

“Hundred-Day Reform” of 1898

KANG You-wei
康有為
(1858-1927)



LIANG Qi-chao
梁啟超
(1873-1929)



TAN Si-tong
譚嗣同
(1865-1898)

南海康有為

凡例

一、凡天下之大，不外義理、制度兩端。義理者何？曰實理，曰公理，曰私理是也。制度者何？曰公法，曰比例之公法、私法是也。實理明則公法定，間有不能定者，則以有益於人道者爲斷，然二者均合衆人之見定之。

一、是書於凡可用實測之理而與制度無關者仍不録，理涉渺茫，無從實測者更不録。

一、是書於地球上諸教所有制度，其非大背實理者，必蒐輯無遺。雖顯背實理，而地球上之人猶有行用者，亦盡輯無遺。必既背實理，又無復有行用之人者，始不登録。其兩教相同之制度，則按語中亦詳言之，此外更參以新得之公法及比例之法。凡一門制度，必取其出自幾何公理及最有益於人道者爲公法，其餘則皆作比例，然亦分別比例之次第焉。其難易分別之處，要皆合衆深明公法之人議定之。

一、凡有憑空擬出一法，欲行則殊不可行者，雖不過欲置爲比例之末，仍不收焉。必雖仍在可行之例者，此書乃修(二)。

Complete Book on Concrete Principles and Postulates [of Human Relationship]

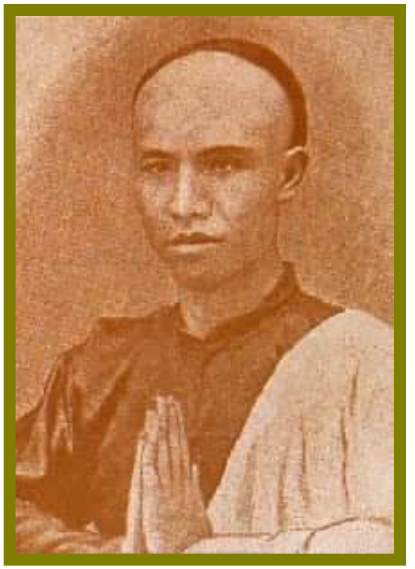
《實理公法全書》，c. 1888; later incorporated into *Book of Great Unity*

《大同書》，1913.

瀏陽譚壯飛先生著

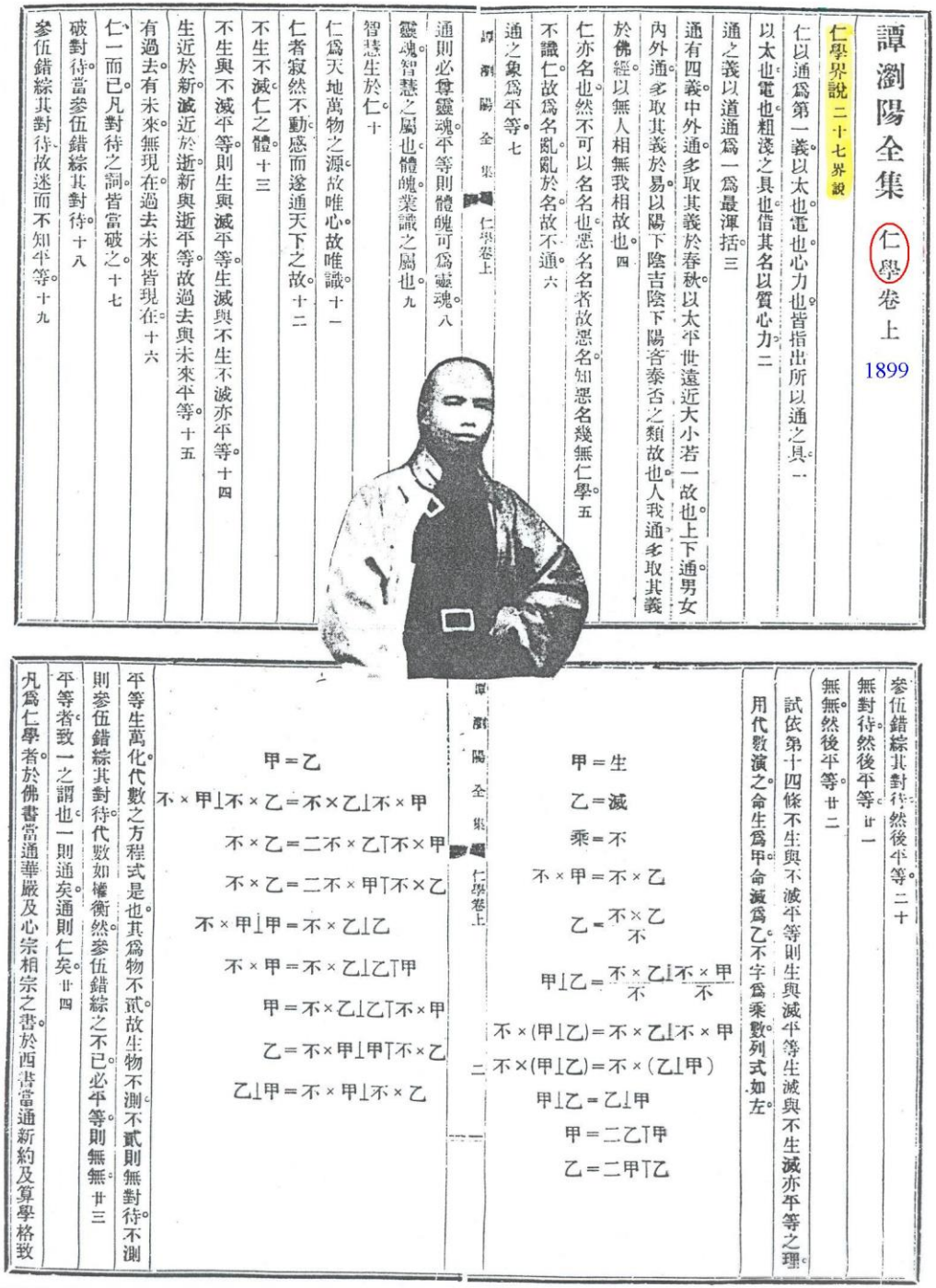
仁學

國民報社藏板



TAN Si-tong
譚嗣同
(1865-1898)

Tan Sitong [譚嗣同]
On Moral Philosophy
《仁學》, (1899)



譚瀏陽全集 仁學卷上 1899

仁學界說二十七界說

仁以通爲第一義以太也電也心力也皆指出所以通之具一

以太也電也粗淺之具也借其名以質心力二

通之義以道通爲一爲最渾括三

通有四義中外通多取其義於春秋以太平世遠近大小若一故也上下通男女內外通多取其義於易以陽下陰吉陰下陽吝泰否之類故也人我通多取其義於佛經以無人相無我相故也四

仁亦名也然不可以名名也惡名名者故惡名知惡名幾無仁學五

不識仁故爲名亂亂於名故不通六

通之象爲平等七

通則必尊靈魂平等則體魄可爲藏魂八

靈魂智慧之屬也體魄業識之屬也九

智慧生於仁十

仁爲天地萬物之源故唯心故唯識十一

仁者寂然不動感而遂通天下之故十二

不生不滅仁之體十三

不生與不滅平等則生與滅平等生滅與不生不滅亦平等十四

生近於新滅近於逝新與逝平等故過去與未來平等十五

有過去有未來無現在過去未來皆現在十六

仁一而已凡對待之詞皆當破之十七

破對待當參伍錯綜其對待十八

參伍錯綜其對待故迷而不知平等十九

參伍錯綜其對待然後平等二十

無對待然後平等廿一

無無然後平等廿二

試依第十四條不生與不滅平等則生與滅平等生滅與不生滅亦平等之理用代數演之命生爲甲命滅爲乙不字爲乘數列式如左

甲＝生

乙＝滅

乘＝不

不×甲＝不×乙

$乙=\frac{不\times乙}{不}$

$甲\perp乙=\frac{不\times乙\perp不\times甲}{不}$

不×(甲⊥乙)＝不×乙⊥不×甲

不×(甲⊥乙)＝不×(乙⊥甲)

甲⊥乙＝乙⊥甲

甲＝二乙⊥甲

乙＝二甲⊥乙

甲＝乙

不×甲⊥乙×不＝乙×不⊥甲×不

不×乙＝二不×乙＝不×甲⊥乙

乙×不⊥甲×不＝二不×甲⊥乙

不×甲⊥乙＝不×甲⊥乙

甲⊥乙＝不×甲＝不×乙⊥乙

甲＝不×乙⊥乙⊥甲

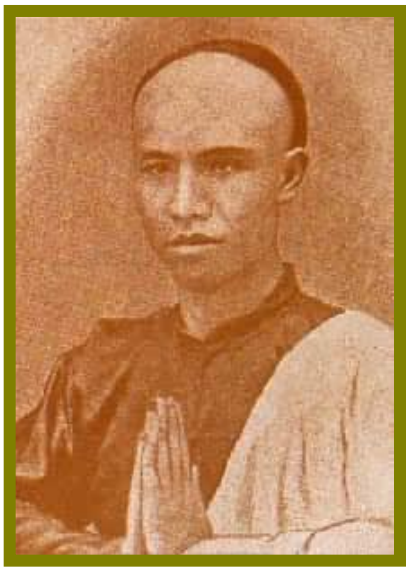
乙＝不×甲⊥甲⊥乙

乙⊥甲＝不×甲⊥乙

平等生萬化代數之方程式是也其爲物不貳故生物不測不貳則無對待不測則參伍錯綜其對待代數如權衡然參伍錯綜之不已必平等則無無廿三

平等者致一之謂也一則通矣通則仁矣廿四

凡爲仁學者於佛書當通華嚴及心宗相宗之書於西書當通新約及算學格致



TAN Si-tong

譚嗣同

(1865-1898)

Tan Sitong [譚嗣同]

On Moral Philosophy

《仁學》, (1899)

瀏陽算學館章程 (1897)

……

一、本館之設，原以培植人材，期臻遠大，并非為諸生謀食計。算學為格致初基， 必欲詣極精微，終身亦不能盡。……

……

一、古者六藝，禮、樂、射、御、書、數，算特其一。即論西人致用，自算學始，不自算學止。諸生所學，當先立乎其大者，重倫常，慎言行、崇禮義，尚廉恥。而於所業則勿忘，勿助長，無欲速，無見小利，知及仁守，富有日新，然後體立用行，推己及物。……

譚嗣同立

"Vision and Mission" of the Liuyang College of Mathematics

Mathematics is the foundation of science.

The study starts with mathematics but does not end with it.

「自明之末葉，利瑪竇等輸入當時所謂西學者於中國，而學問研究方法上，生一種外來的變化。其初惟治天算者宗之，後則漸應用於他學。」

梁啟超
(1873-1929)

梁啟超,《清代學術概論》
(原刊於《改造雜誌》, 1920, 1921.)

「夫歐幾里得之書，條理統系，精密絕倫，非僅論數論象之書，實為希臘民族精神之所表現。此滿文譯本及數理精蘊本皆經刪改，意在取便實施，而不知轉以是失其精意。」

陳寅恪 (1890-1969)

陳寅恪
《幾何原本》滿文譯本跋
1931

「此書為益，能令學理者祛其浮氣，練其精心，學事者資其定法，發其巧思，故舉世無一人不當學。」

徐光啟
(1562-1633)

XU Guang-qi, *Various Reflections on Jihe Yuanben*
(徐光啟,《幾何原本雜議》) 1607

此書有五不可學。

躁心人不可學。

麤心人不可學。

滿心人不可學。

妬心人不可學。

傲心人不可學。

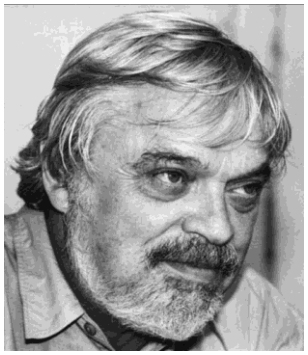
故學此者不止增才。

亦德基也。

徐光啟
(1562-1633)

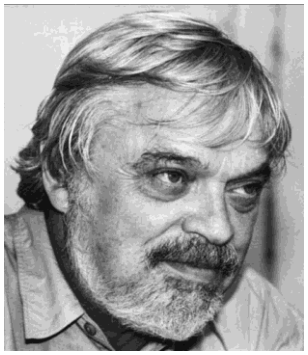
徐光啟，《幾何原本雜議》(1607)

"Geometry is a phenomenon of the human culture. ... Geometry, as well as mathematics in general, helps in moral and ethical education of children. ... Geometry develops mathematical intuition, introduces a person to independent mathematical creativity. ... Geometry is a point of minimum for the distance between school mathematics and the mathematics of high level."



Igor Fedorovich Sharygin (沙雷金)
(1937-2004)

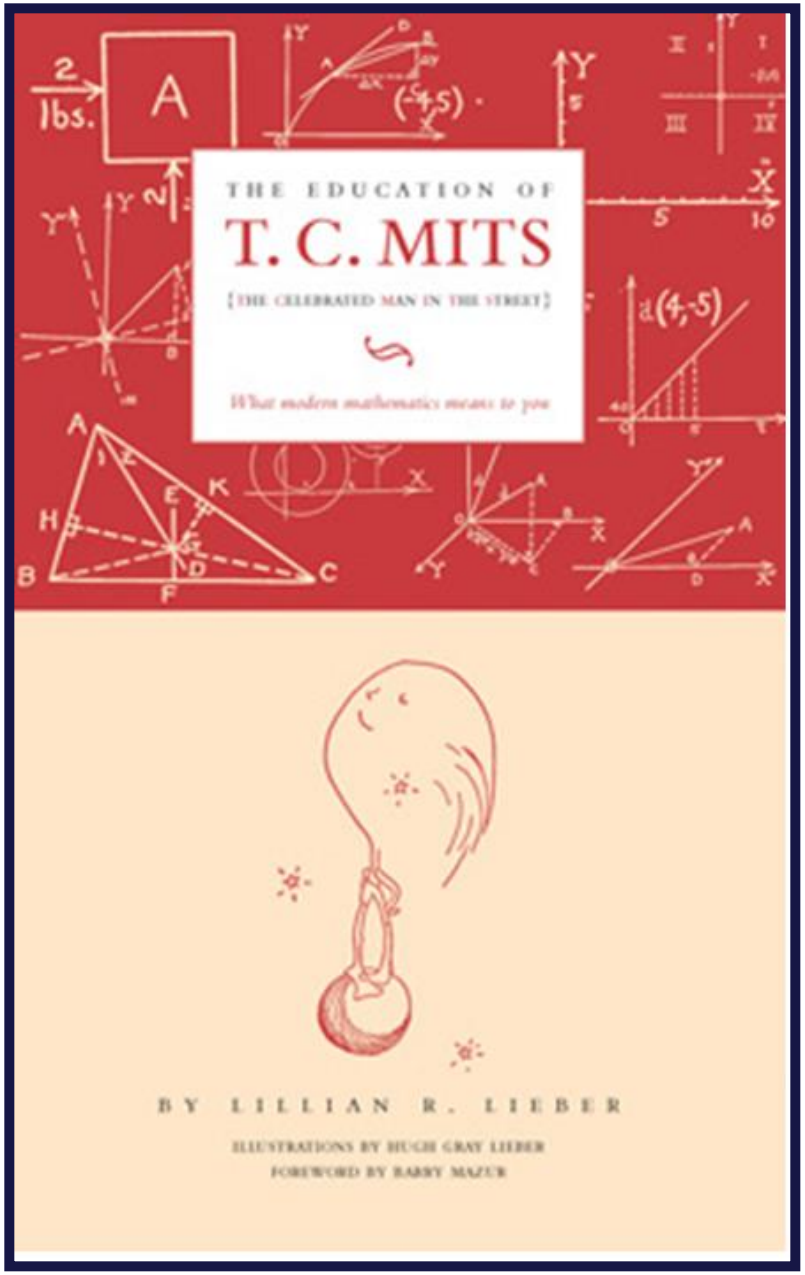
“Learning mathematics builds up our virtues, sharpens our sense of justice and our dignity, strengthens our innate honesty and our principles. The life of mathematical society is based on the idea of proof, one of the most highly moral ideas in the world .”



Igor Fedorovich Sharygin (沙雷金)
(1937-2004)

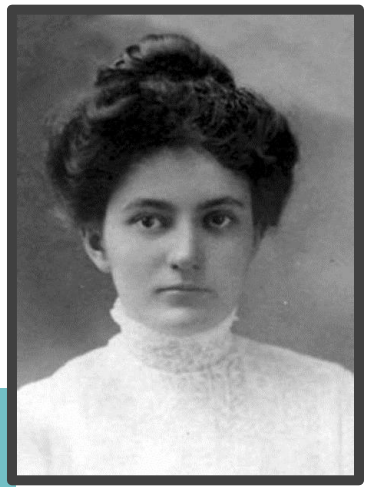
“Rigour is to the mathematician what morality is to man.
(嚴謹之於數學家，
猶如道德之於
一般人。)”

André Weil
(1906-1998)



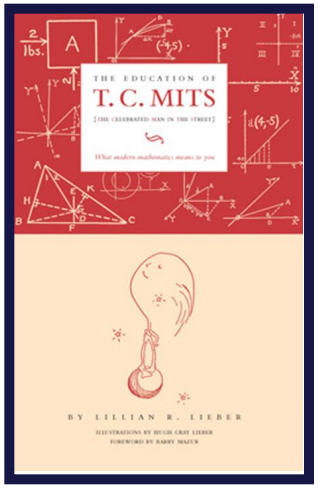
Lillian R. Lieber, *The Education of T.C. Mits: What Modern Mathematics Means to You*, originally published in 1942; republished in 2007.

[T.C. Mits = The Celebrated Man In The Street]



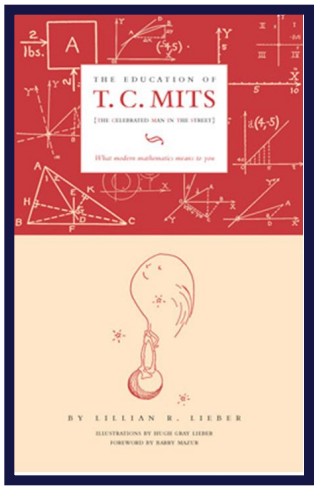
Lillian Rosanoff Lieber, (1886-1986)

And so you see how
Mathematics can throw light
on various subjects
which many people discuss
glibly and carelessly
since they have never been trained
to examine ideas
With that METICULOUS CARE
With which a mathematician works.



Lillian R. Lieber, *The Education of T.C. Mits: What Modern Mathematics Means to You*, originally published in 1942; republished in 2007.

There is a model for straight thinking
which we all MUST try to imitate.
This is not the
noisy argumentation of
the pseudo-thinkers.
Rather it is
quiet,
honest,
careful,
COMPETENT.



The Moral: **Do not be NAÏVE — Use the methods of Mathematics.**

Lillian R. Lieber, *The Education of T.C. Mits: What Modern Mathematics Means to You*, originally published in 1942; republished in 2007.