# The Power and "*Way*" of Mathematics

## Part I

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The Chinese University of Hong Kong

**Most people, even if they do not study mathematics, will recognize the utility of mathematics and its power in solving problems in various areas and in explaining what is happening in the world around us, not just the physical world but also the biological and the economic-financial world as well.**

**What endows mathematics**
**With its power and utility?**
**What is the "*Way*" of mathematics?**
**How did mathematics develop**
**into a subject as it is today?**

**What is the nature of the subject?**
**Is mathematics a tool,**
**or a way of thinking,**
**or a part of culture?**

**Is a mathematical proof a kind of ritual observed by a certain sect (called mathematicians)?**

**What is a proof for — verification, or enhancement of understanding, or training of the mind, or only for professional conscience?**

**Why are we so certain that a theorem in mathematics really holds true?**

# How do mathematicians work?

**Do they only do complicated calculations? Do they just follow logical deduction, or do they allow wild thinking that may not yet be justified according to logic?**

**When a mathematician talks about the beauty and elegance of mathematics, what is meant by that?**

**Why can mathematics,**
**abstract and seemingly**
**man-made though it is,**
**explain and be applicable to**
**so many different phenomena**
**in the real world?**
**Do we discover the mathematics,**
**or**
**do we invent the mathematics?**

**It would take at least a full course to just touch upon such issues, which soon become philosophical. Since one can hardly do justice to such issues in one or two lectures, we will in these two lectures (October 15 and 22) try only to have a glimpse of the power and "*Way*" of mathematics by going through some selected examples. We will work through and discuss these examples in class together, presuming knowledge in school mathematics. In the second lecture we will talk more about proofs in mathematics.**

# Course YSCN0002
# Mathematics: A Cultural Heritage
# 數學：文化的傳承

This course attempts to elaborate an exhortation of Hermann Weyl through examples gathered from the long history of mathematics, around our daily lives, in other areas of human endeavour and in Nature. Rather than transmitting a body of technical knowledge in mathematics the emphasis is placed on appreciating, contemplating and discussing about the beauty, the utility and the "*Way*" of mathematics.


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**From 1999 to 2009 it was offered as an elective for students in any faculty and department at HKU.**

"Anschauung" of Mathematics
數學觀
Sense of
數學意識
Mathematics
數學之「道」
The "Way" of Mathematics
數學之美
Beauty of Mathematics
數學之用
Utility of Mathematics
Knowledge in Mathematics
數學知識

Nigcradoc ot a seerchar ta na Elnshig nevstriyiu, ti sedon't tamert ni thaw redor het stetler ni a rowd rae, het loyn torptamin hingt si hatt rifst nad salt telter ear ta het girth caple. Het sert anc eb a lotta sems nad uoy anc tills dear ti thowitu belmorp. Hist si cabusee ew od ton dear revey tetler by sitfle tub het rowd sa a howle.

Aoccdrnig to a rseerach at an Elingsh uinervtisy, it deosn't mttaer in waht oredr the ltteers in a wrod are, the olny iprmoatnt tihng is taht frist and lsat ltteer are at the rghit pclae. The rset can be a toatl mses and you can sitll raed it wouthit porbelm. Tihs is bcuseae we do not raed ervey lteter by itslef but the wrod as a wlohe.

According to a research at an English university, it doesn't matter in what order the letters in a word are, the only important thing is that first and last letter are at the right place. The rest can be a total mess and you can still read it without problem. This is because we do not read every letter by itself but the word as a whole.

According to a research at an English university, it doesn't matter in what order the letters in a word are, the only important thing is that first and last letter are at the right place. The rest can be a total mess and you can still read it without problem. This is because we do not read every letter by itself but the word as a whole.

To what extent do you believe in the claim made in this internet meme that went the rounds in September of 2003?

The reason for showing you
this experiment is not to discuss with you
the topic of cognition in reading.

I just want to borrow this example as
an analogy to indicate that in order
to **understand** and **appreciate**
mathematics it is not imperative that
you should follow every single step
of the technical content
in detail meticulously.

You need only get an impression of how
things **hang together as a whole**.

孫子算經序

孫子曰夫算者天地之經緯羣生之元首五常之本末陰陽之父母星辰之建號三光之表裏五行之準平四時之終始萬物之祖宗六藝之綱紀稽羣倫之聚散考二氣之降升推寒暑之迭運步遠近之殊同觀天道精微之兆基察地理從横之長短采神祇之所在極成敗之符驗窮道德之理究性命之情立規矩準方圓謹法度約尺丈立權衡平重輕剖毫釐析黍絫歷億載而不朽施八極而無疆散之不可勝究斂之不盈掌握嚮之者富有餘背之者貧且寠心開者幼沖而即悟意閑者皓首而難精夫欲學之者必務量能揆己志在所專如是則焉有不成者哉

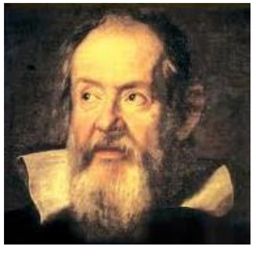
孫子曰：夫算者，天地之經緯，群生之元首；五常之本末，陰陽之父母；星辰之建號，三光之表裏；五行之準平，四時之終始；萬物之祖宗，六藝之綱紀。

稽群倫之聚散，考二氣之降升；推寒暑之迭運，步遠近之殊同；觀天道精微之兆基，察地理從横之長短；采神祇之所在，極成敗之符驗；窮道德之理，究性命之情。立規矩，準方圓，謹法度，約尺丈，立權衡，平重輕，剖毫釐，析黍絫；

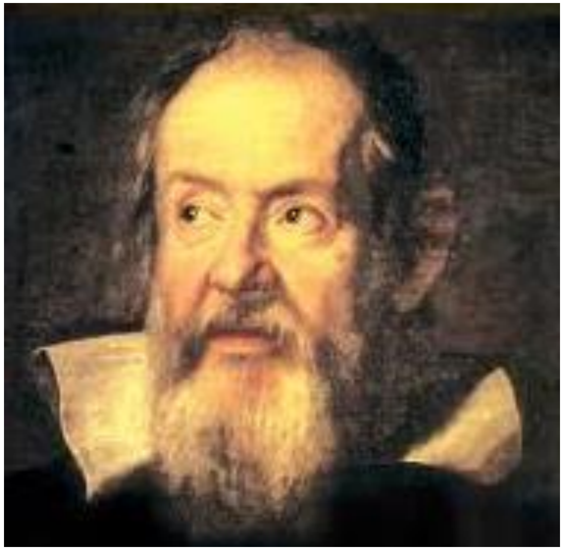
歷億載而不朽，施八極而無疆。散之不可勝究，斂之不盈掌握。嚮之者富有餘，背之者貧且寠；心開者幼沖而即悟，意閑者皓首而難精。夫欲學之者必務量能揆己，志在所專。如是則焉有不成者哉。

《孫子算經》(公元四/五世紀)

*"Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and reads the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth."*



Il Saggiatore (The Assayer)
Letter to the Illustrious and Very Reverend Don Virginio Cesarini from Galileo Galilei (1623)



Galileo Galilei
(1564-1642)

**HOW (MUCH)** rather than **WHY**?

Power of Mathematics

**[a quantitative rather than a qualitative description]**

**Curiosity**

**Imagination**

**Disciplined** and **Critical Thinking**

(precision in **mathematics** as well as in **words**)

# The Su-do-ku Craze

明報 (Ming Pao) 2005.05.16

英報形容如「21世紀扭計骰」

前港法官掀歐美數字拼圖熱

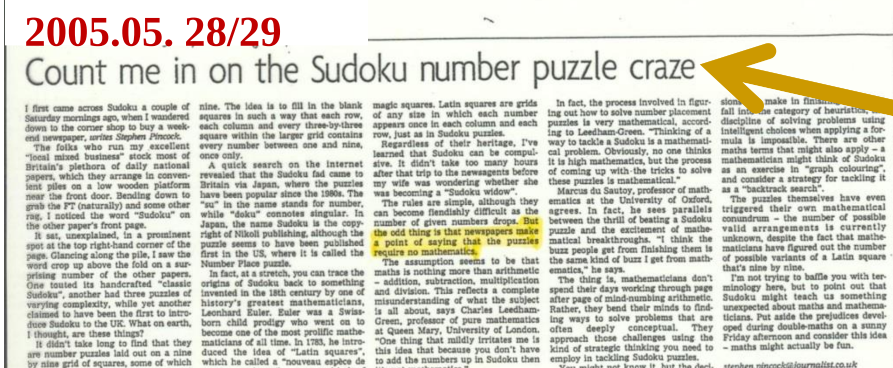
「數字拼圖」遊戲玩法

智力遊戲迷 一見傾情努力研究

日人改良 遊戲重見天日

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 1 | | | | 8 | | |
| | 7 | | 3 | 1 | | | 9 | |
| 3 | | | | 4 | 5 | | | 7 |
| | 9 | | 7 | | | 5 | | |
| | 4 | 2 | | 5 | | 1 | 3 | |
| | | 3 | | | 9 | | 4 | |
| 2 | | | 5 | 7 | | | | 4 |
| | 3 | | | 9 | 1 | | 6 | |
| | | 4 | | | | 3 | | |

Financial Times 2005.05. 28/29



Count me in on the Sudoku number puzzle craze

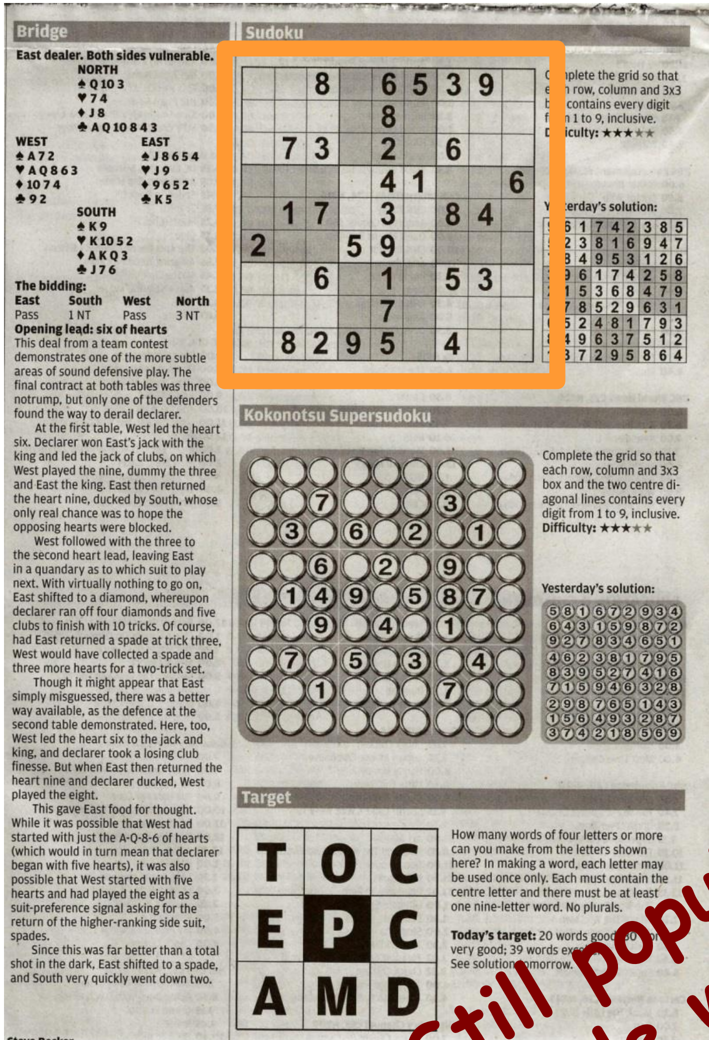
**Count me in on the Sudoku number craze**

Stephen Pincock

*Financial Times*

May 28, 2005.

# Sudoku in Puzzles' Corners of newspapers



## Bridge

East dealer. Both sides vulnerable.

NORTH ♠Q 10 3 ♥7 4 ♦J 8 ♣A Q 10 8 4 3

WEST ♠A 7 2 ♥A Q 8 6 3 ♦10 7 4 ♣9 2

EAST ♠J 8 6 5 4 ♥J 9 ♦9 6 5 2 ♣K 5

SOUTH ♠K 9 ♥K 10 5 2 ♦A K Q 3 ♣J 7 6

The bidding:

| East | South | West | North |
|---|---|---|---|
| Pass | 1 NT | Pass | 3 NT |

**Opening lead: six of hearts**

This deal from a team contest demonstrates one of the more subtle areas of sound defensive play. The final contract at both tables was three notrump, but only one of the defenders found the way to derail declarer.

At the first table, West led the heart six. Declarer won East's jack with the king and led the jack of clubs, on which West played the nine, dummy the three and East the king. East then returned the heart nine, ducked by South, whose only real chance was to hope the opposing hearts were blocked.

West followed with the three to the second heart lead, leaving East in a quandary as to which suit to play next. With virtually nothing to go on, East shifted to a diamond, whereupon declarer ran off four diamonds and five clubs to finish with 10 tricks. Of course, had East returned a spade at trick three, West would have collected a spade and three more hearts for a two-trick set.

Though it might appear that East simply misguessed, there was a better way available, as the defence at the second table demonstrated. Here, too, West led the heart six to the jack and king, and declarer took a losing club finesse. But when East then returned the heart nine and declarer ducked, West played the eight.

This gave East food for thought. While it was possible that West had started with just the A-Q-8-6 of hearts (which would in turn mean that declarer began with five hearts), it was also possible that West started with five hearts and had played the eight as a suit-preference signal asking for the return of the higher-ranking side suit, spades.

Since this was far better than a total shot in the dark, East shifted to a spade, and South very quickly went down two.

Steve Becker

## Sudoku

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 8 | | 6 | 5 | 3 | 9 | |
| | | | | 8 | | | | |
| | 7 | 3 | | 2 | | 6 | | |
| | | | | 4 | 1 | | | 6 |
| | 1 | 7 | | 3 | | 8 | 4 | |
| 2 | | | 5 | 9 | | | | |
| | | 6 | | 1 | | 5 | 3 | |
| | | | | 7 | | | | |
| | 8 | 2 | 9 | 5 | | 4 | | |

Complete the grid so that each row, column and 3x3 box contains every digit from 1 to 9, inclusive.
Difficulty: ★★★★★

Yesterday's solution:

## Kokonotsu Supersudoku

Complete the grid so that each row, column and 3x3 box and the two centre diagonal lines contains every digit from 1 to 9, inclusive.
Difficulty: ★★★★★

Yesterday's solution:

## Target

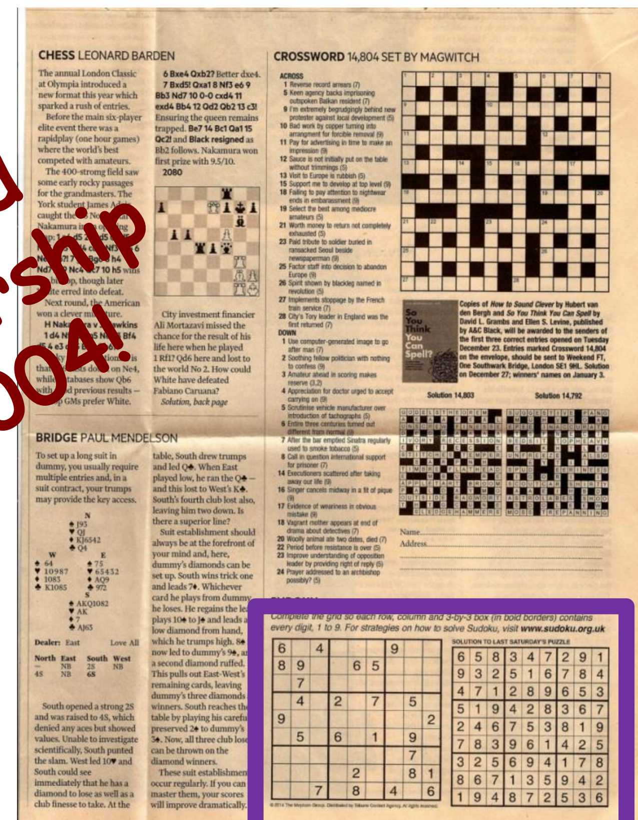
| | | |
|---|---|---|
| T | O | C |
| E | P | C |
| A | M | D |

How many words of four letters or more can you make from the letters shown here? In making a word, each letter may be used once only. Each must contain the centre letter and there must be at least one nine-letter word. No plurals.

**Today's target:** 20 words good; 30 words very good; 39 words excellent. See solution tomorrow.

South China Morning Post, May 21, 2015

Financial Times Asian edition, 2014



CHESS LEONARD BARDEN

CROSSWORD 14,804 SET BY MAGWITCH

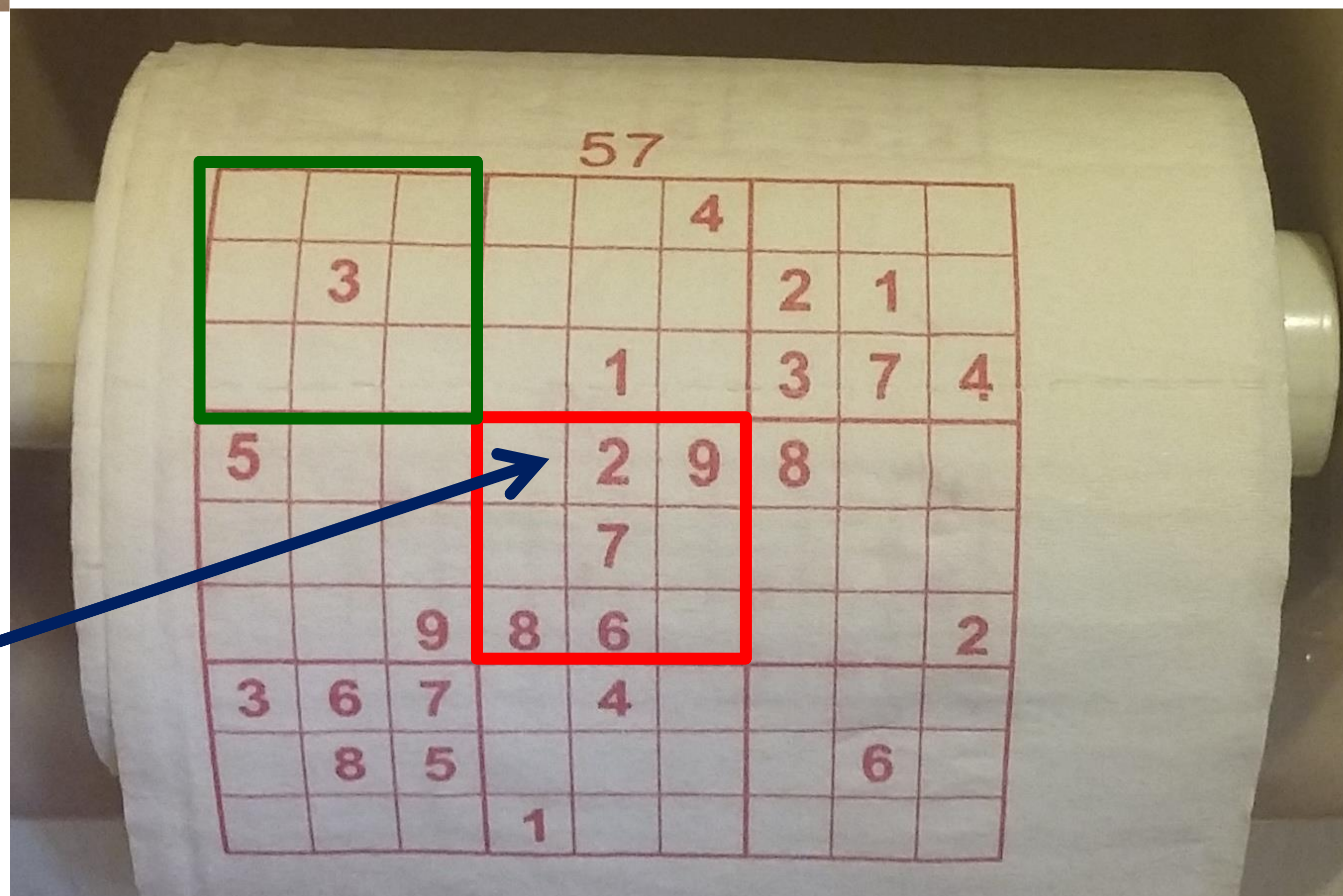
BRIDGE PAUL MENDELSON

Complete the grid so each row, column and 3-by-3 box (in bold borders) contains every digit, 1 to 9. For strategies on how to solve Sudoku, visit www.sudoku.org.uk

SOLUTION TO LAST SATURDAY'S PUZZLE

Still popular after the puzzle was introduced to the public readership a decade ago in 2004!

# Sudoku in Toilet




This cell can only be filled with 3 or 4, because 1 and 5 are not allowed.

**Count me in. The real puzzle behind Sudoku is the idea that maths doesn't come into play. Well, the number crunchers will have the last laugh.**

**Stephen Pincock**

***Financial Times*, May 28, 2005.**

**"One thing that mildly irritates me is this idea that because you don't have to add the numbers up in Sudoku then it's not mathematics."**

**Charles Leedham-Green,**
**Professor of Pure Mathematics**
**at Queen Mary College, University of London.**

**P = PHILOSOPHY MASTER**
**J = MONSIEUR JOURDAIN**

**Moliere's *Le Bourgeois gentilhomme* (October 14, 1670)**

**Molière**
**[Jean-Baptiste Poquelin]**
**(1622-1673)**

**P : What isn't verse is prose, and what's not prose is verse.**

**J : And this, the way I speak. What name would be applied to the --**

**P : The way you speak?**

**J : Yes.**

**P : Prose.**

**J : It's prose?**

**P : Decidedly.**

**J : Oh really? So when I say: "Nicole bring me my slippers and fetch my nightcap," Is that prose?**

**P : Most clearly.**

**J: Well, what do you know about that ! These forty years now, I've been speaking in prose without knowing it ! How grateful am I to you for teaching me that ! …**

**Sudoku, as a puzzle of pure logical deduction, provides a good means to study the underlying mental processes in naïve individuals, that is, those who have had no training in logic.**

**How much ability in pure deductive reasoning do naïve individuals really possess?**

**N.Y. Louis Lee, G.P. Goodwin, P.N. Johnson-Laird, The psychological puzzle of Soduku, *Thinking & Reasoning*, 14(4) (2008), 342-364.**

- **Logically naïve individuals have the competence to make deductions about abstract matters, and they enjoy exercising this ability.**

- **The teaching of mathematics can be more effective when it focuses on abstract  matters rather than on concrete everyday examples**

**N.Y. Louis Lee, G.P. Goodwin, P.N. Johnson-Laird, The psychological puzzle of Soduku, *Thinking & Reasoning*, 14(4) (2008), 342-364.**

**As Piaget recognised, our ability to make deductions about abstract matters remote from our mundane life is a fundamental human characteristic, and one that is essential to intellectual progress.**

**N.Y. Louis Lee, G.P. Goodwin, P.N. Johnson-Laird, The psychological puzzle of Soduku, *Thinking & Reasoning*, 14(4) (2008), 342-364.**

# Good points about mathematical games and puzzles

- **Nurturing of**
  - observation
  - concentration
  - patience
  - curiosity
  - flexible thinking
- **Training of**
  - logical thinking
  - space visualization
  - systematic analysis
  - meticulous working

**Genuine "learning in pleasure [愉快學習]"**

嬰兒生無石師而能言
與能言者處也。 莊子・外物（公元前四世紀）

**[ When a child is born, it needs no great teacher ;**
**nevertheless it learns to talk as it lives with those who talk. ]**

**Zhuangzi (Chuang Tzu)**
***Book 26 : Affected from Outside***
**(4th century B.C.E.)**

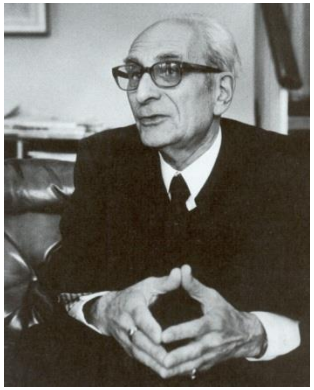
**We don't stop playing because we grow old; we grow old because we stop playing.**

**(Variably attributed to Benjamin Franklin, Oliver Wendell Holmes (Sr. or Jr.), Herbert Spencer, George Bernard Shaw)**

**"Groos [Karl Groos] well says that children are young because they play, and not vice versa; and he might have added, men grow old because they stop playing, and not conversely, for play is, at bottom, growth, and at the top of the intellectual scale it is the eternal type of research from sheer love of truth."**

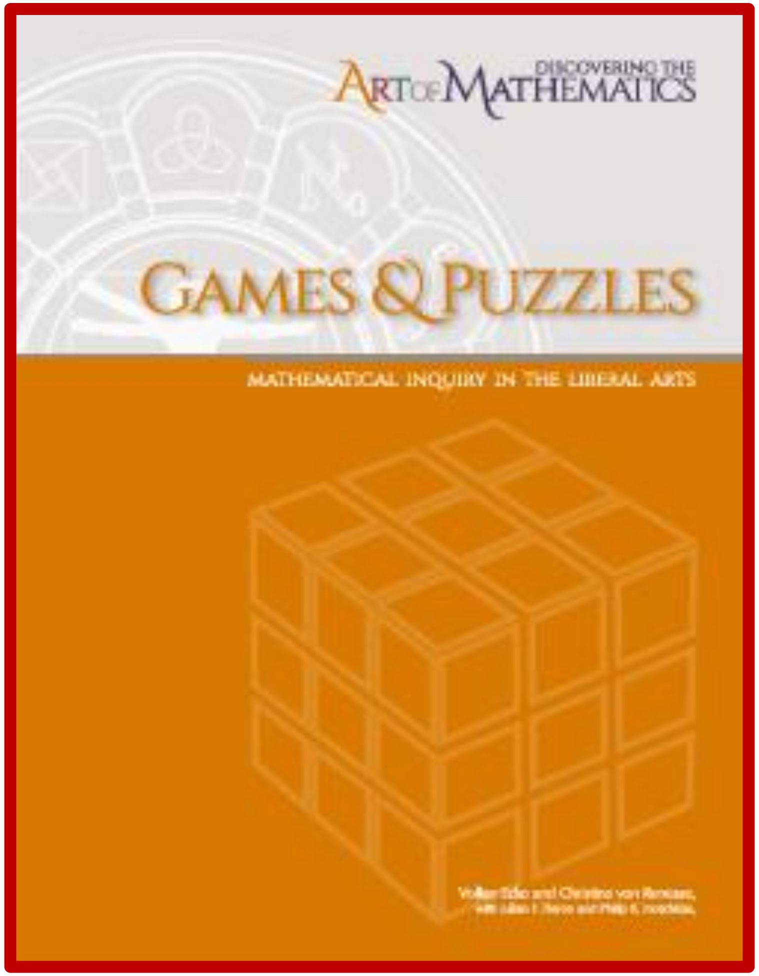
**G. Stanley Hall, *Adolescence: Its Psychology and its Relations to Physiology, Anthropology, Sociology, Sex, Crime, Religion and Education* (1904)**

**" ... the student who chooses the teaching profession does not bid farewell to the world of childhood: on the contrary he is trying to remain within it."**



**要保持童心！**

**Claude Lévi-Strauss**
**(1908 - 2009)**

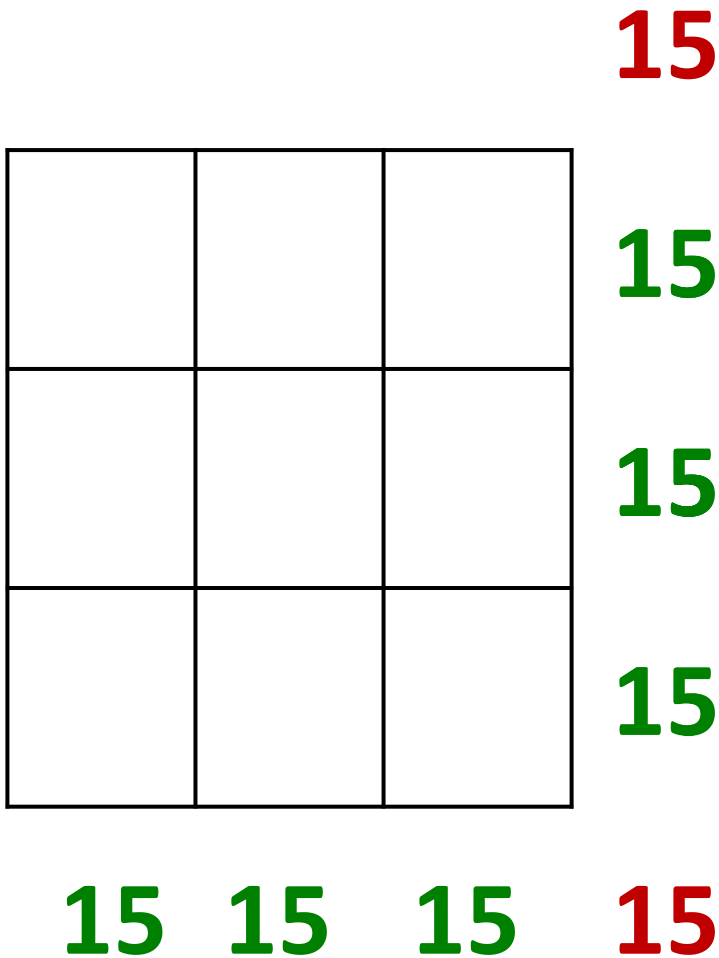



# Games and Puzzles

Volker Ecke and Christine von Renesse, with Julian F. Fleron and Philip K. Hotchkiss,

2015 ; current version 2018 .

## Section 3.2 : Radon/Kaczmarz Puzzles




You are given the three row sums, three column sums and the two diagonal sums, fill in the nine cells in the 3 x 3 grid with suitable positive integers chosen from 1 to 9.

| | | | |
|---|---|---|---|
| | | | 15 |
| 2 | 7 | 6 | 15 |
| 9 | 5 | 1 | 15 |
| 4 | 3 | 8 | 15 |
| 15 | 15 | 15 | 15 |

You are given the three row sums, three column sums and the two diagonal sums, fill in the nine cells in the 3 x 3 grid with suitable positive integers chosen from 1 to 9.

| | | | |
|---|---|---|---|
| | | | 15 |
| 5 | 5 | 5 | 15 |
| 5 | 5 | 5 | 15 |
| 5 | 5 | 5 | 15 |
| 15 | 15 | 15 | 15 |

**You are given the three row sums, three column sums and the two diagonal sums, fill in the nine cells in the 3 x 3 grid with suitable positive integers chosen from 1 to 9.**

| | | | 15 |
|---|---|---|---|
| 4 | 6 | 5 | 15 |
| 6 | 5 | 4 | 15 |
| 5 | 4 | 6 | 15 |
| 15 | 15 | 15 | 15 |

**You are given the three row sums, three column sums and the two diagonal sums, fill in the nine cells in the 3 x 3 grid with suitable positive integers chosen from 1 to 9.**

| | | | 15 |
|---|---|---|---|
| 3 | 5 | 7 | 15 |
| 9 | 5 | 1 | 15 |
| 3 | 5 | 7 | 15 |
| 15 | 15 | 15 | 15 |

You are given the three row sums, three column sums and the two diagonal sums, fill in the nine cells in the 3 x 3 grid with suitable positive integers chosen from 1 to 9.

| | | | |
|---|---|---|---|
| | | | 15 |
| 7 | 2 | 6 | 15 |
| 4 | 5 | 6 | 15 |
| 4 | 8 | 3 | 15 |
| 15 | 15 | 15 | 15 |

You are given the three row sums, three column sums and the two diagonal sums, fill in the nine cells in the 3 x 3 grid with suitable positive integers chosen from 1 to 9.

| | | |
|---|---|---|
| 2 | 7 | 6 |
| 9 | 5 | 1 |
| 4 | 3 | 8 |

| | | |
|---|---|---|
| 5 | 5 | 5 |
| 5 | 5 | 5 |
| 5 | 5 | 5 |

| | | |
|---|---|---|
| 4 | 6 | 5 |
| 6 | 5 | 4 |
| 5 | 4 | 6 |

| | | |
|---|---|---|
| 3 | 7 | 5 |
| 7 | 5 | 3 |
| 5 | 3 | 7 |

| | | |
|---|---|---|
| 2 | 8 | 5 |
| 8 | 5 | 2 |
| 5 | 2 | 8 |

| | | |
|---|---|---|
| 1 | 9 | 5 |
| 9 | 5 | 1 |
| 5 | 1 | 9 |

| | | |
|---|---|---|
| 3 | 5 | 7 |
| 9 | 5 | 1 |
| 3 | 5 | 7 |

| | | |
|---|---|---|
| 4 | 5 | 6 |
| 7 | 5 | 3 |
| 4 | 5 | 6 |

| | | |
|---|---|---|
| 3 | 6 | 6 |
| 8 | 5 | 2 |
| 4 | 4 | 7 |

**There are altogether 41 solutions to the puzzle, falling into essentially 9 types with the remaining ones obtained via rotation or reflection.**

| | | |
|---|---|---|
| 3 | 9 | 3 |
| 5 | 5 | 5 |
| 7 | 1 | 7 |

| | | |
|---|---|---|
| 6 | 6 | 3 |
| 2 | 5 | 8 |
| 7 | 4 | 4 |

| | | |
|---|---|---|
| 4 | 8 | 3 |
| 4 | 5 | 6 |
| 7 | 2 | 6 |

| | | |
|---|---|---|
| 7 | 5 | 3 |
| 1 | 5 | 9 |
| 7 | 5 | 3 |

| | | |
|---|---|---|
| 5 | 7 | 3 |
| 3 | 5 | 7 |
| 7 | 3 | 5 |

There are **9 unknowns** and **12 equations**. Why can't the given conditions pin down the solution?

**Even if in addition we are given the five *NW-SE* diagonal sums the answer is still not unique.**

| | | |
|---|---|---|
| 3 | 9 | 3 |
| 5 | 5 | 5 |
| 7 | 1 | 7 |

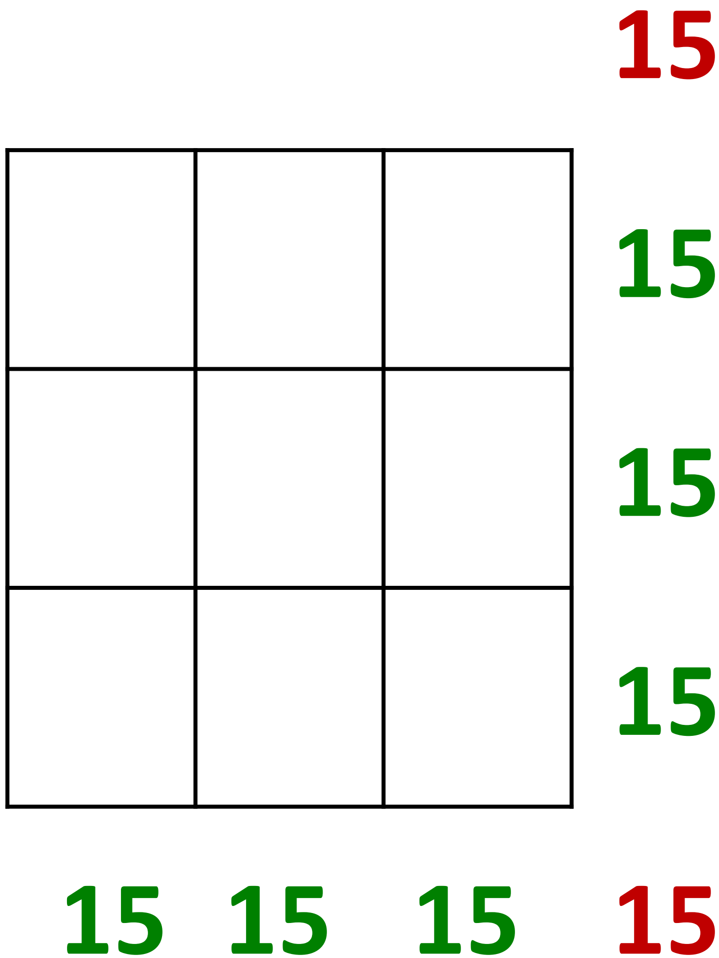
| | | |
|---|---|---|
| 6 | 6 | 3 |
| 2 | 5 | 8 |
| 7 | 4 | 4 |

| | | |
|---|---|---|
| 4 | 8 | 3 |
| 4 | 5 | 6 |
| 7 | 2 | 6 |

| | | |
|---|---|---|
| 7 | 5 | 3 |
| 1 | 5 | 9 |
| 7 | 5 | 3 |

| | | |
|---|---|---|
| 5 | 7 | 3 |
| 3 | 5 | 7 |
| 7 | 3 | 5 |

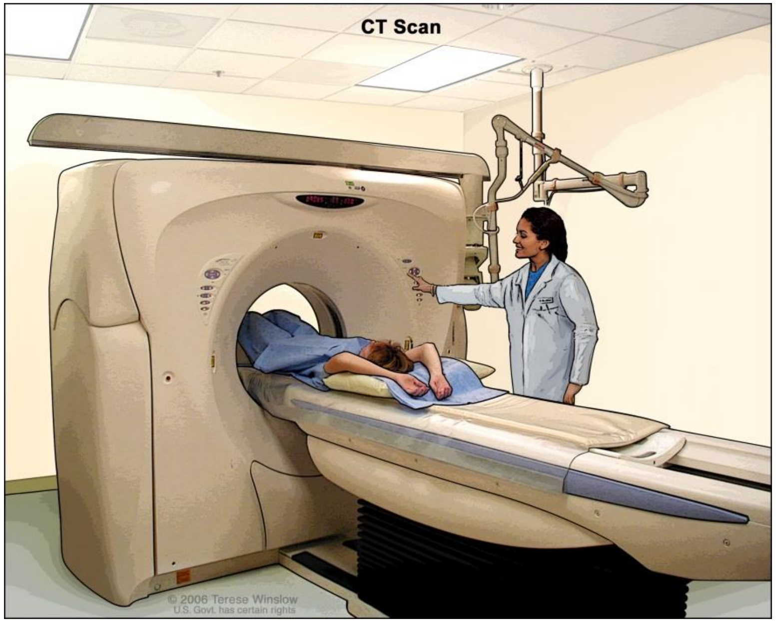
**If in addition we are given all the ten diagonal sums, then the answer will be unique.**

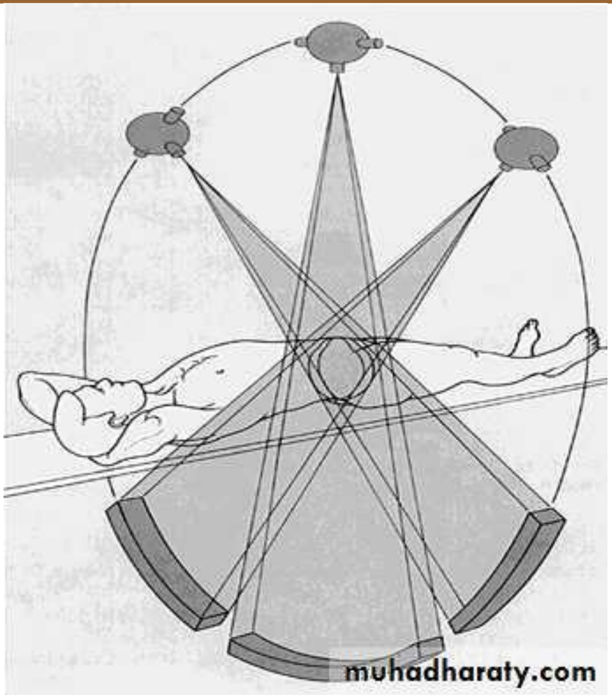


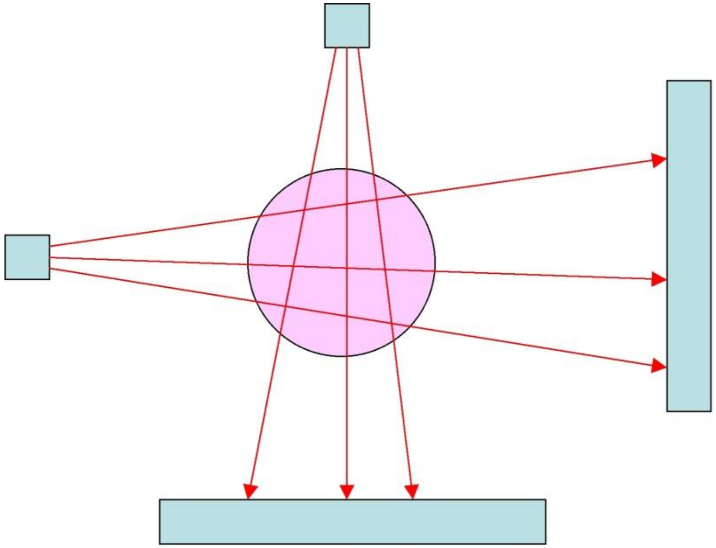

In general we are looking at a system of 8 linear equations with 9 unknowns, the rank of the coefficient matrix being equal to 7. Of the 9 unknowns there are 7 pivotal unknowns and 2 free unknowns.

**Question:** What sort of conditions will guarantee a unique solution if one exists?

# CT Scan (Computerized Tomography)



**Basically we try to figure out the entries of a large grid knowing the row sums, column sums, diagonal sums, etc.**

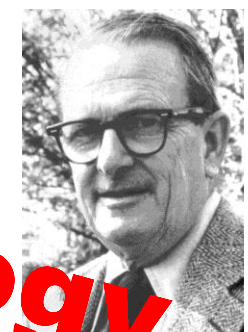
Kaczmarz's algorithm
for solving a system of
linear equations, 1937.

Mathematics,

Radon (Inverse) Transform,
1917.

Stefan Kaczmarz
(1895-1940)

Johann Radon
(1887-1950)



Science, Technology,
Engineering,

Nobel Prize for Physiology or
Medicine for development of
diagnostic technique of X-ray
CT (computed tomography), 1979.

Godfrey Newbold
Hounsfield
(1919-2004)

Allan McLeod
Cormack
(1924-1998)

$$\cos(\pi/4) = 0.7071067\ldots$$

707 (1957)

717

727

737

747

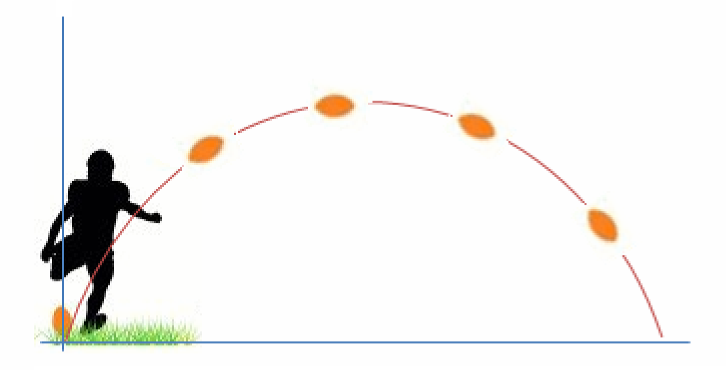
757

767

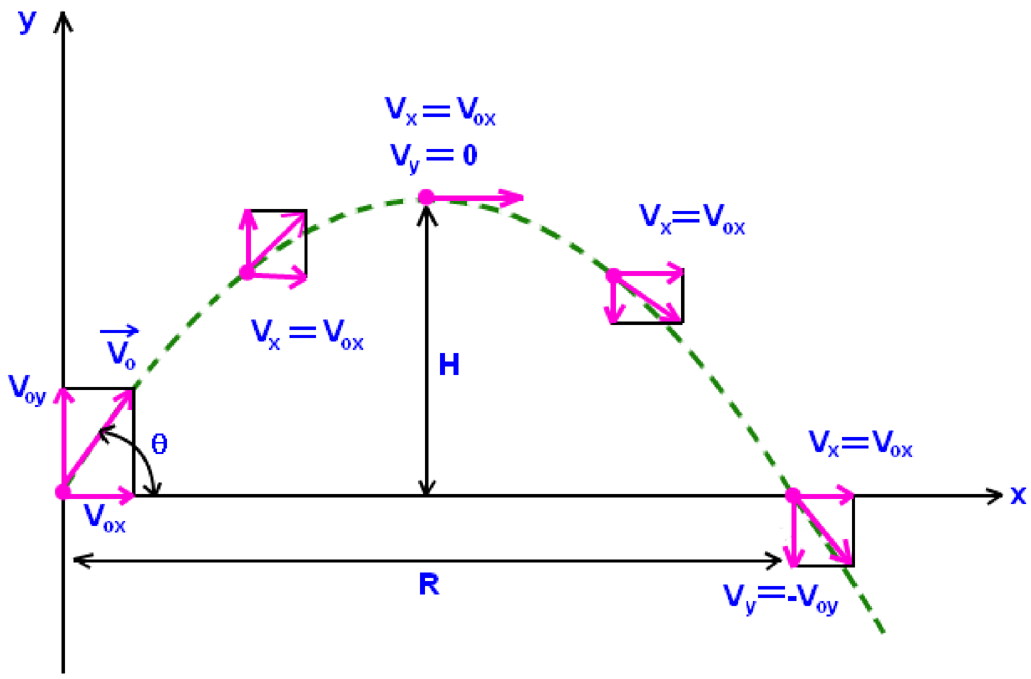
777

787

# Boeing Airplanes



# Projectile 拋體運動




http://ggbtu.be/m1082291

## SMET

**S**cience
**M**athematics
**E**ngineering
**T**echnology

It was said that in 2001 Judith Ramaley, then Assistant Director for Education and Human Resources at NSF, thought that SMET does not sound as good as STEM, so she changed the acronym to STEM!

## STEM

**S**cience
**T**echnology
**E**ngineering
**M**athematics

| STEM | STEAM |
|---|---|
| **S**cience | **S**cience |
| **T**echnology | **T**echnology |
| **E**ngineering | **E**ngineering |
| **M**athematics | **A**rts |
| | **M**athematics |

| STEM | STREAM |
| --- | --- |
| Science | Science |
| Technology | Technology |
| Engineering | Reading |
| Mathematics | Engineering |
| | Arts |
| | Mathematics |

## STEM

**S**cience

**T**echnology

**E**ngineering

**M**athematics

## iSTREAM

***i**nformation science*

**S**cience

**T**echnology

**R**eading

**E**ngineering

**A**rts

**M**athematics

## STEM

Science

Technology

Engineering

Mathematics

## STREAiM

Science

Technology

Reading

Engineering

Artificial

intelligence

Mathematics

With a humanistic and caring mind breed a feeling of humbleness and tolerance by learning from the long history of the human race. Science is not almighty to command everything. Instead, we should learn how to live in harmony with Mother Nature and with others.

**THAMES**

**T**echnology

**H**umanities

**A**rts/**A**rtificial intelligence

**M**athematics

**E**ngineering

**S**cience

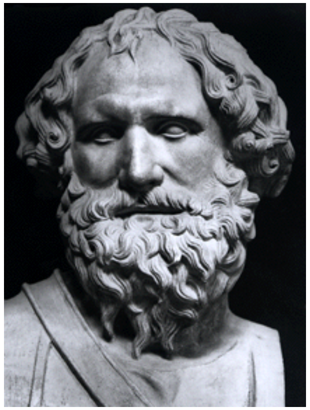
「依我看，STEM 不是一個學科，也更加不僅是一堆現代科技產品。

STEM 是一種綜合意識，滲透在不同的學科，以學習和運用數學知識及科學知識，再以工程手段配合現代科技，改善生活。

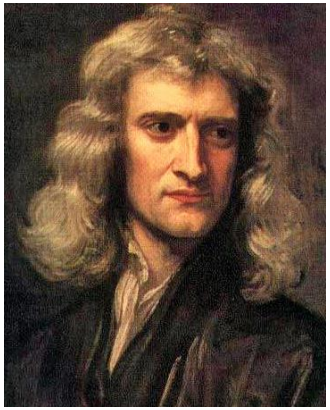
STEM 蘊含一種探索精神及思想方式，揉合了數學思維及科學精神，通過實驗、觀察、理論整理以尋求知識，進而創新。」

蕭文強,推薦序,
盧安迪,《STEM教育與美國》(2018), 頁10.

# STEM
# ANEG



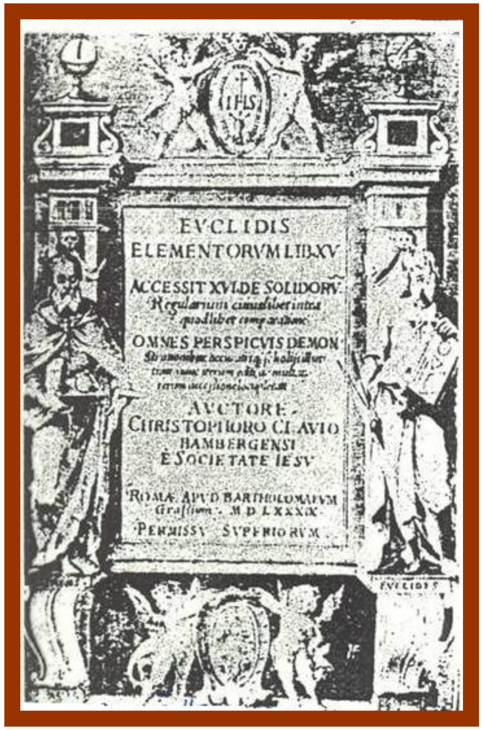
Archimedes
(287–212 B.C.E.)



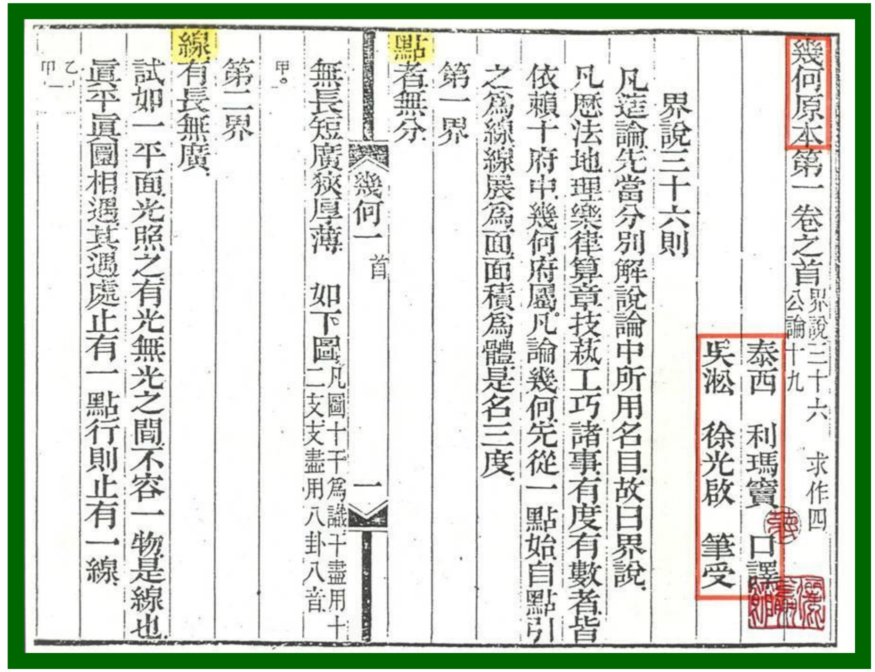
Isaac Newton
(1642-1727)

Leonhard Euler
(1707-1783)

Carl Friedrich Gauss
(1777-1855)




**C. CLAVIUS, *EUCLIDIS ELEMENTORUM LIBRI XV***
**(1574; 1589)**
**EUCLID'S *ELEMENTS***
**(c. 300 B.C.E.)**


幾何原本第一卷之首 界說三十六 求作四 公論十九
泰西 利瑪竇 口譯
吳淞 徐光啟 筆受
界說三十六則
凡造論先當分別解說論中所用名目故曰界說
凡歷法地理樂律算章技藝工巧諸事有度有數者皆
依賴十府中幾何府屬凡論幾何先從一點始自點引
之為線線展為面面積為體是名三度
第一界
點者無分
幾何一 首
無長短廣狹厚薄 如下圖 凡圖十干為識干盡用十二支支盡用八卦八音
甲。
第二界
線有長無廣
試如一平面光照之有光無光之間不容一物是線也
真平真圓相遇其遇處止有一點行則止有一線
乙 甲

# 幾何原本

***Jihe Yuanben* [幾何原本]**
**translated by Matteo Ricci**
**and XU Guang-qi (1607)**

**Matteo Ricci**
**(1552-1610)**

# Chinese translation of *Elements* by Matteo Ricci [利瑪竇] and XU Guang-qi [徐光啟] in 1607

**XU Guang-qi**
**(1562-1633)**

「度數旁通十事」：

「其一（天氣），其二（測量），其三（樂律），
其四（軍事），其五（會計），其六（建築），
其七（機械），其八（輿圖），其九（醫學），
其十（時計）。

STEM in early seventeenth century China

「右十條於民事似為關切。
臣聞之周髀算經云：禹之所以
治天下者，句股之所繇生也。
蓋凡物有形有質，莫不資於
度數故耳。」

徐光啟，條議曆法修正歲差疏, 1629

徐光啟
XU Guang-qi
(1562-1633)

# What endows mathematics with its power and utility?

— Logic?

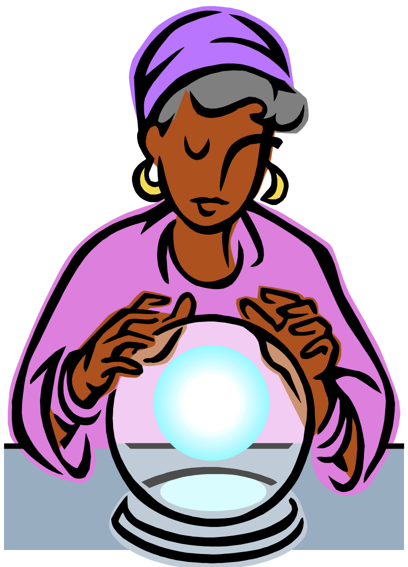
— Computation?

— Use of symbols?

**Why would many people feel uneasy about the use of symbols?**

1) Choose any two-digit number you like.
2) Take the sum of the two digits.
3) Subtract the sum from the chosen number.
4) Find the icon in the list corresponding to the answer you obtain in step (3). Then click the crystal ball.

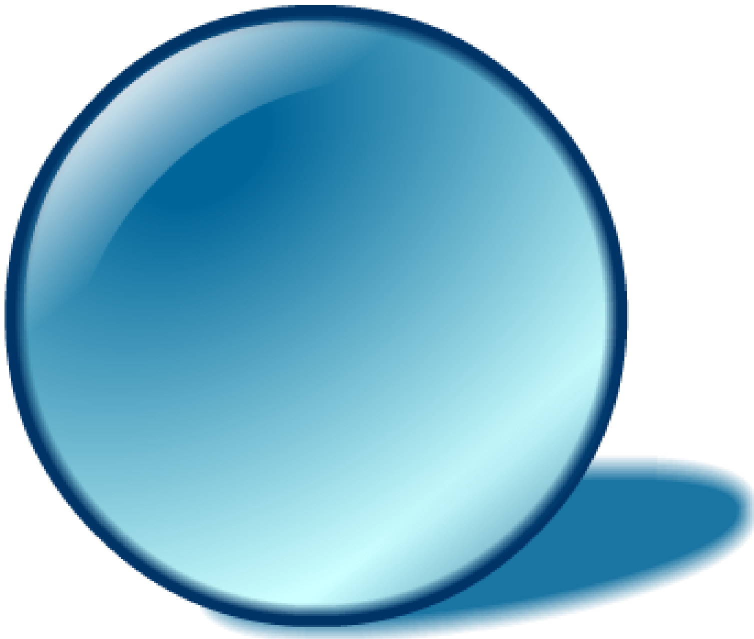
**The crystal ball can read your mind by revealing the correct icon!**



**Example:**

$$53 - (5 + 3) = 53 - 8 = 45$$

# The Flash Mind Reader



| | | | | |
|---|---|---|---|---|
| 99 ☪ | 79 ♉ | 59 ♍ | 39 ☯ | 19 🔔 |
| 98 ❄ | 78 ♌ | 58 ☪ | 38 ♑ | 18 ♋ |
| 97 ♑ | 77 ♉ | 57 ❍ | 37 ♎ | 17 ☼ |
| 96 □ | 76 ♓ | 56 ♐ | 36 ♋ | 16 ⚐ |
| 95 ♋ | 75 ♒ | 55 ♑ | 35 ♍ | 15 □ |
| 94 ✞ | 74 ☪ | 54 ♋ | 34 ⚐ | 14 🔔 |
| 93 ♌ | 73 ♉ | 53 ♓ | 33 ♓ | 13 ♋ |
| 92 ⚐ | 72 ♋ | 52 ✞ | 32 🔔 | 12 ☺ |
| 91 ☪ | 71 ♒ | 51 ♉ | 31 ⚐ | 11 ☺ |
| 90 ☪ | 70 ✞ | 50 ♉ | 30 ❄ | 10 📖 |
| 89 ♑ | 69 ♐ | 49 ☪ | 29 ♍ | 9 ♋ |
| 88 ☯ | 68 ♑ | 48 ♌ | 28 ♑ | 8 ♉ |
| 87 ☪ | 67 ♒ | 47 ⚐ | 27 ♋ | 7 □ |
| 86 ♐ | 66 ♍ | 46 ♏ | 26 ♌ | 6 ⚐ |
| 85 ☼ | 65 ⚐ | 45 ♋ | 25 ☺ | 5 ❁ |
| 84 ⚐ | 64 □ | 44 ♌ | 24 ♑ | 4 ❍ |
| 83 ♌ | 63 ♋ | 43 ❄ | 23 ♒ | 3 ⚐ |
| 82 ♍ | 62 ♎ | 42 ❍ | 22 ❁ | 2 ♎ |
| 81 ♋ | 61 📖 | 41 ☼ | 21 ♌ | 1 ☪ |
| 80 ♉ | 60 ✡ | 40 ♒ | 20 ♒ | 0 ☪ |

Choose any two digit number, add together both digits and then subtract the total from your original number.*

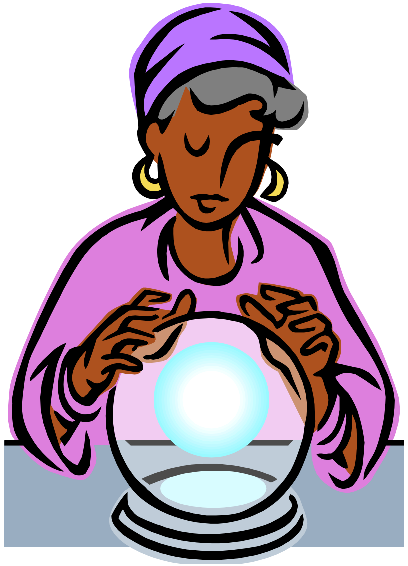
When you have the final number look it up on the chart and find the relevant symbol. Concentrate on the symbol and when you have it clearly in your mind click on the crystal ball and it will show you the symbol you are thinking of...

* For example if you chose 23: 2+3 = 5. 23 minus 5 will give you your answer.

created by Andy Naughton

1) Choose any two-digit number you like.
2) Take the sum of the two digits.
3) Subtract the sum from the chosen number.
4) Find the icon in the list corresponding to the answer you obtain in step (3). Then click the crystal ball.

**The crystal ball can read your mind by revealing the correct icon!**



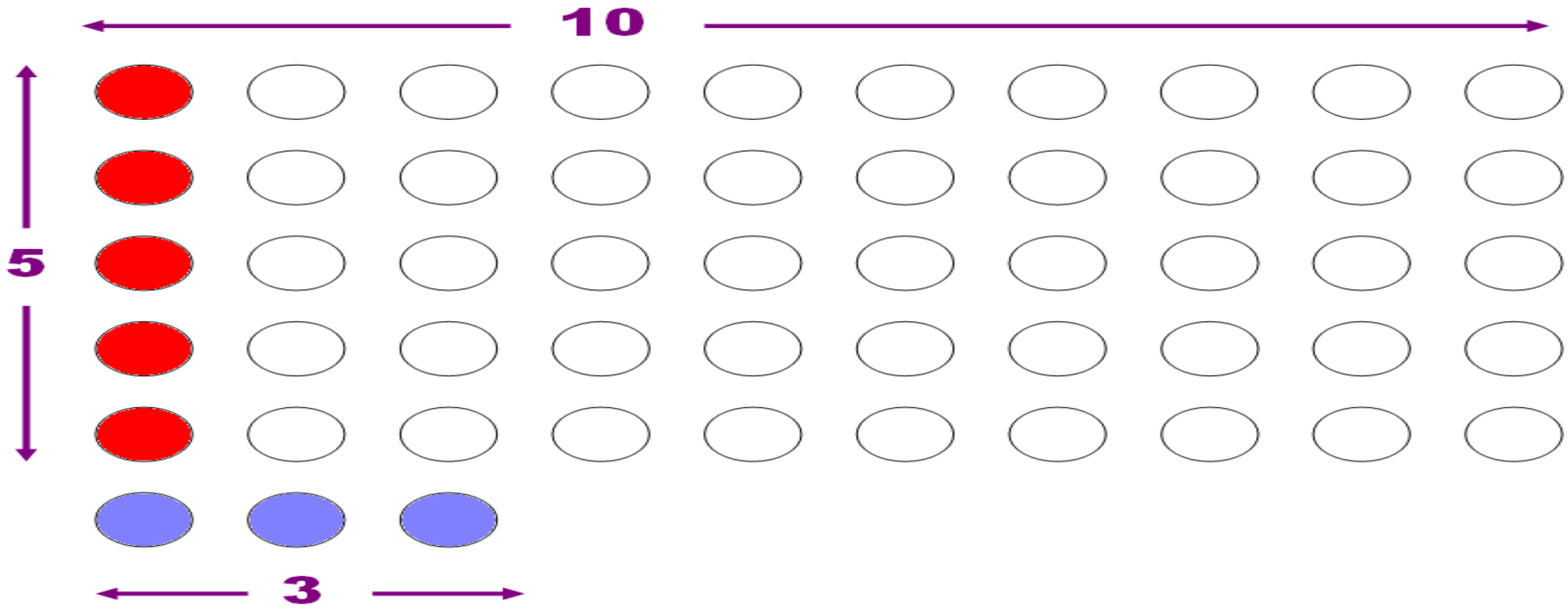
**Example:**

$$53 - (5 + 3) = 53 - 8 = 45$$

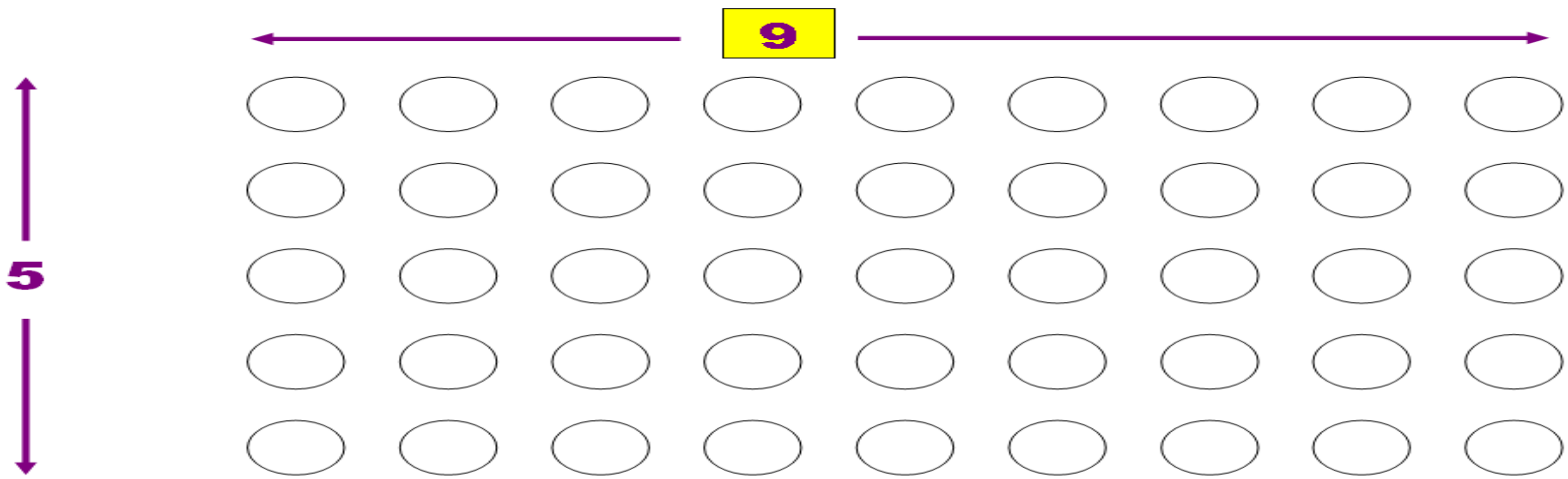
$$N = 10A + B$$

$$N - (A + B) = 10A + B - A - B = 9A$$

$$53 - (5 + 3) = 9 \times 5$$



$$53 - (5 + 3) = 9 \times 5$$



**SYMBOL ANXIETY**
（符號焦慮症）

**SYMBOL POWER**
（符號的威力）

**Rhetoric Algebra**（言辭代數） → **Syncoptic Algebra**（略寫代數） → **Symbolic Algebra**（符號代數）

(G.H.F. Nesselmann, 1842)

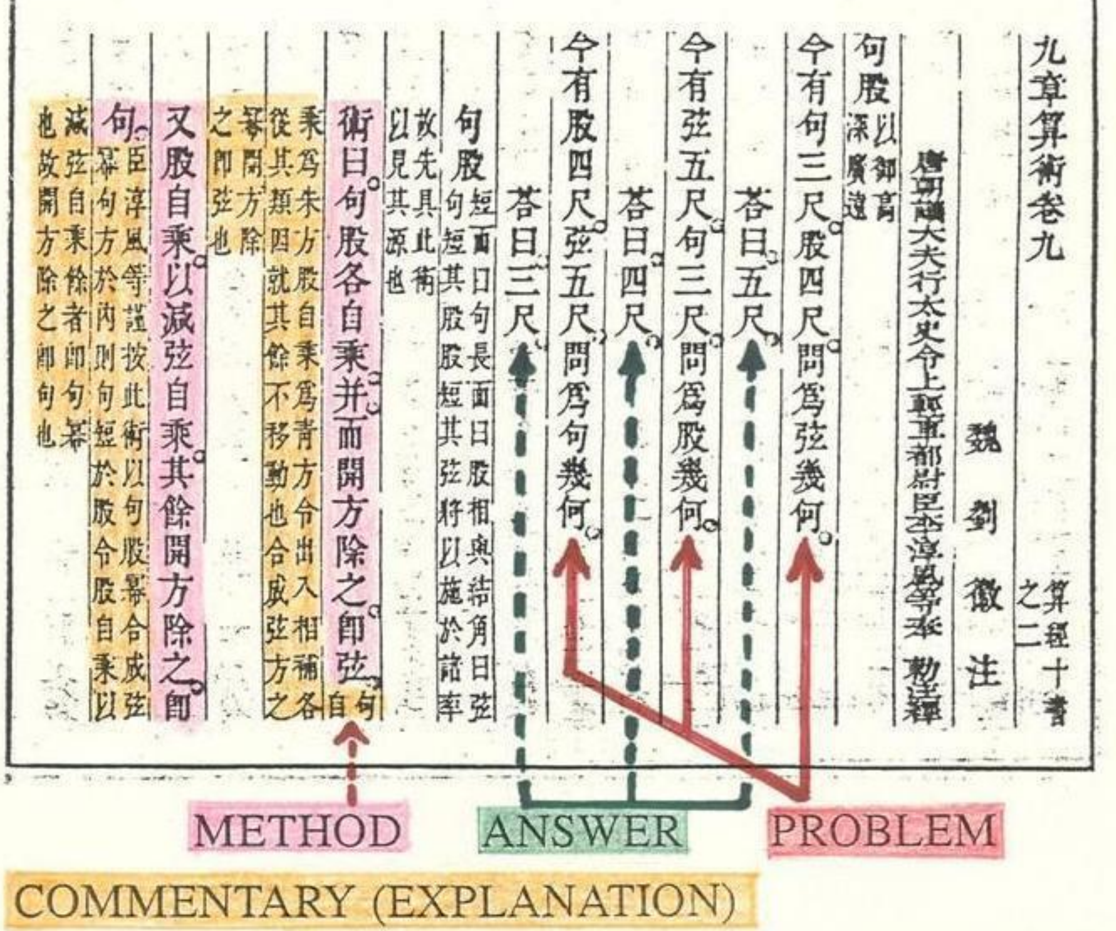
*Jiuzhang Suanshu* [九章算術 **Nine Chapters on the Mathematical Art**]

***ca.* 100 B.C.E. — 100 C.E.**

**Commentary on *Jiiuzhang Suanshu***

**by LIU Hui [劉徽]**

## Chapter 9 : *Gougu* [勾股]


九章算術卷九　算經十書之二

魏　劉徽　注

唐朝議大夫行太史令上輕車都尉臣李淳風等奉　勅注釋

句股　以御高深廣遠

今有句三尺，股四尺，問為弦幾何。

荅曰：五尺。

今有弦五尺，句三尺，問為股幾何。

荅曰：四尺。

今有股四尺，弦五尺，問為句幾何。

荅曰：三尺。

句股　短面曰句，長面曰股，相與結角曰弦。句短其股，股短其弦。將以施於諸率，故先具此術以見其源也。

術曰：句股各自乘，并而開方除之，即弦。句自乘為朱方，股自乘為青方，令出入相補，各從其類，因就其餘不移動也，合成弦方之冪。開方除之，即弦也。

又股自乘，以減弦自乘，其餘開方除之，即句。臣淳風等謹按：此術以句股冪合成弦冪，句方於內，則句短於股。令股自乘，以減弦自乘，餘者即句冪也。故開方除之，即句也。



# RHETORIC

# Diophantus, *Arithmetica* (*c*. 250 C.E.)

$3z^2+12z+9$
= a square

$3z^2+12z+9$
$=(mz-3)^2$

$z=\dfrac{6m+12}{m^2-3}$

Τετάχθω ὁ μὲν ἐλάχιστος Μ̊ δ̄, ἡ δὲ τοῦ μέσου πλ. ς ᾱ Μ̊ β̄· αὐτὸς ἄρα ἔσται ὁ □ος, ΔΥ ᾱ ς δ̄ Μ̊ δ̄.

Ἐπεὶ οὖν ἡ ὑπεροχὴ τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχίστου γον μέρος ἐστίν, καὶ ἔστιν ἡ ὑπεροχὴ τοῦ μέσου καὶ τοῦ ἐλαχίστου ΔΥ ᾱ ς δ̄, ὥστε ἡ ὑπεροχὴ τοῦ μεγίστου καὶ τοῦ μέσου ἔσται ΔΥ γχ ς ᾱ γχ· καὶ ἔστιν ὁ μέσος ΔΥ ᾱ ς δ̄ Μ̊ δ̄· ὁ ἄρα μέγιστος ἔσται ΔΥ ᾱ γχ ς ε̄ γχ Μ̊ δ̄ ἴσ. □ῳ· πάντα θκις· ΔΥ ἄρα ιβ̄ ς μη̄ Μ̊ λϛ̄ ἴσ. □ῳ καὶ τὸ δον αὐτῶν· ΔΥ γ̄ ς ιβ̄ Μ̊ θ̄ ἴσ. □ῳ.

Ἔτι δὲ θέλω τὸν μέσον τετράγωνον ἐλάσσονα εἶναι Μ̊ ιϛ̄, καὶ τὴν πλ. δηλαδὴ ἐλάσσονος Μ̊ δ̄. ἡ δὲ πλευρὰ τοῦ μέσου ἐστὶν ς ᾱ Μ̊ β̄· ἐλάττονές εἰσι Μ̊ δ̄. καὶ κοινῶν ἀφαιρεθεισῶν τῶν β̄ Μ̊, ὁ ς ἔσται ἐλάσσονος Μ̊ β̄.

Γέγονεν οὖν μοι ΔΥ γ̄ ς ιβ̄ Μ̊ θ̄ ἴσ. ποιῆσαι □ῳ. πλάσσω □ὸν τινα ἀπὸ Μ̊ γ̄ λειπουσῶν ς τινας· καὶ γίνεται ὁ ς ἔκ τινος ἀριθμοῦ ϛκις γενομένου καὶ προσλαβόντος τὸν ιβ̄, τουτέστι τῆς ἰσώσεως τῆς ς ιβ̄, καὶ μερισθέντος εἰς τὴν ὑπεροχὴν ᾗ ὑπερέχει ὁ ἀπὸ τοῦ ἀριθμοῦ □ος τῶν ΔΥ τῶν ἐν τῇ ἰσώσει γ̄. ἀπῆκται οὖν μοι εἰς τὸ εὑρεῖν τινα ἀριθμόν, ὃς ϛκις γενόμενος καὶ προσλαβὼν Μ̊ ιβ̄ καὶ μεριζόμενος εἰς τὴν ὑπεροχὴν ᾗ ὑπερέχει ὁ ἀπὸ τοῦ αὐτοῦ □ος τριάδος, ποιεῖ τὴν παραβολὴν ἐλάσσονος Μ̊ β̄.

**SYNCOPTIC**

# 1591 François Viète

published *In Artem Analyticem Isagoge* [Introduction to the Analytic Art]

- **logistica numerosa**
- **logistica speciosa**

**Quod est, Nullum non problema solvere. [There is no problem that cannot be solved.]**

沒有問題是解決不了的！

François Viète (1540-1603)

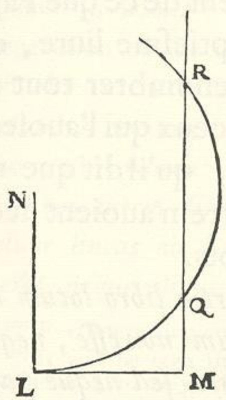
# *La géométrie* (1637)



angle, iuſques a O, en ſorte qu'N O ſoit eſgale a N L, la toute O M eſt $z$ la ligne cherchée. Et elle s'exprime en cete ſorte

$$z \infty \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}.$$

Que ſi iay $yy \infty - ay + bb$, & qu'$y$ ſoit la quantité qu'il faut trouuer, ie fais le meſme triangle rectangle N L M, & de ſa baze M N i'oſte N P eſgale a N L, & le reſte P M eſt $y$ la racine cherchée. De façon que iay $y \infty -\frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$. Et tout de meſme ſi i'auois $x^4 \infty -ax^2 + b^2$. P M ſeroit $x^2$. & i'aurois $x \infty \sqrt{-\frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}}$: & ainſi des autres.




Enfin ſi i'ay $z^2 \infty az - bb$: ie fais N L eſgale à $\frac{1}{2}a$, & L M eſgale à $b$ cõme deuãt, puis, au lieu de ioindre les poins M N, ie tire M Q R parallele a L N. & du centre N par L ayant deſcrit vn cercle qui la couppe aux poins Q & R, la ligne cherchée $z$ eſt M Q, oubiẽ M R, car en ce cas elle s'exprime en deux façons, a ſçauoir $z \infty \frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$, & $z \infty \frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb}$.

Et ſi le cercle, qui ayant ſon centre au point N, paſſe par le point L, ne coupe ny ne touche la ligne droite M Q R, il n'y a aucune racine en l'Equation, de façon qu'on peut aſſurer que la conſtruction du probleſme propoſé eſt impoſſible.

Au

$$z = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2}$$

**SYMBOLIC**


**René Descartes (1596-1650)**

# 1707 Isaac Newton published

*Arithmetica Universalis*
[Universal Arithmetic]
— work done in 1665-1666


**Isaac Newton (1643-1727)**

# Example 1

$$(X-a)(X-b) = X^2 - (a+b)X + ab$$

$$(X-a)(X-b)(X-c)$$
$$= X^3 - (a+b+c)X^2 + (ab+bc+ca)X - abc$$

etc

$$(X-a_1)(X-a_2)\cdots(X-a_n)$$
$$= X^n + c_{n-1}X^{n-1} + \cdots + c_1 X + c_0$$

→ easy multiplication

←

**Does every equation of degree *n* have *n* roots?**

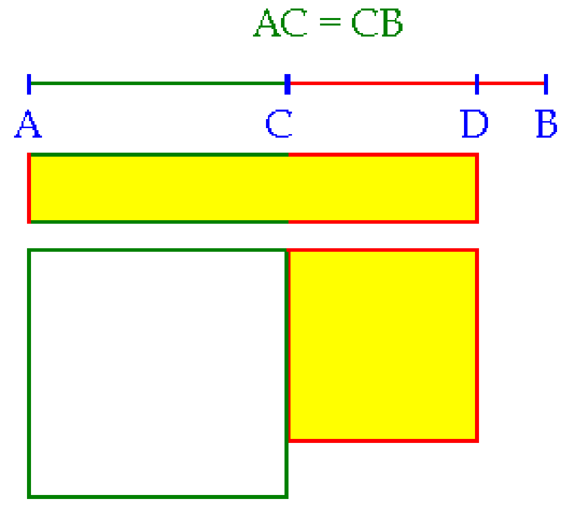
**YES !**

**This result is known as the Fundamental Theorem of Algebra, which was referred to in the works of a number of mathematicians, from A. Girard (1629) to L. Euler (1742), J. L. R. D'Alembert (1748) and C. F. Gauss (1799).**

# Example 2

## Euclid's *Elements*, Book II, Proposition 5 (c. 300 B.C.E.)

**The rectangle contained by AD, DB together with the square on CD is equal to the square on AC.**




In symbols, , $(a - b)(a + b) + b^2 = a^2$,
or $(a - b)(a + b) = a^2 - b^2$.

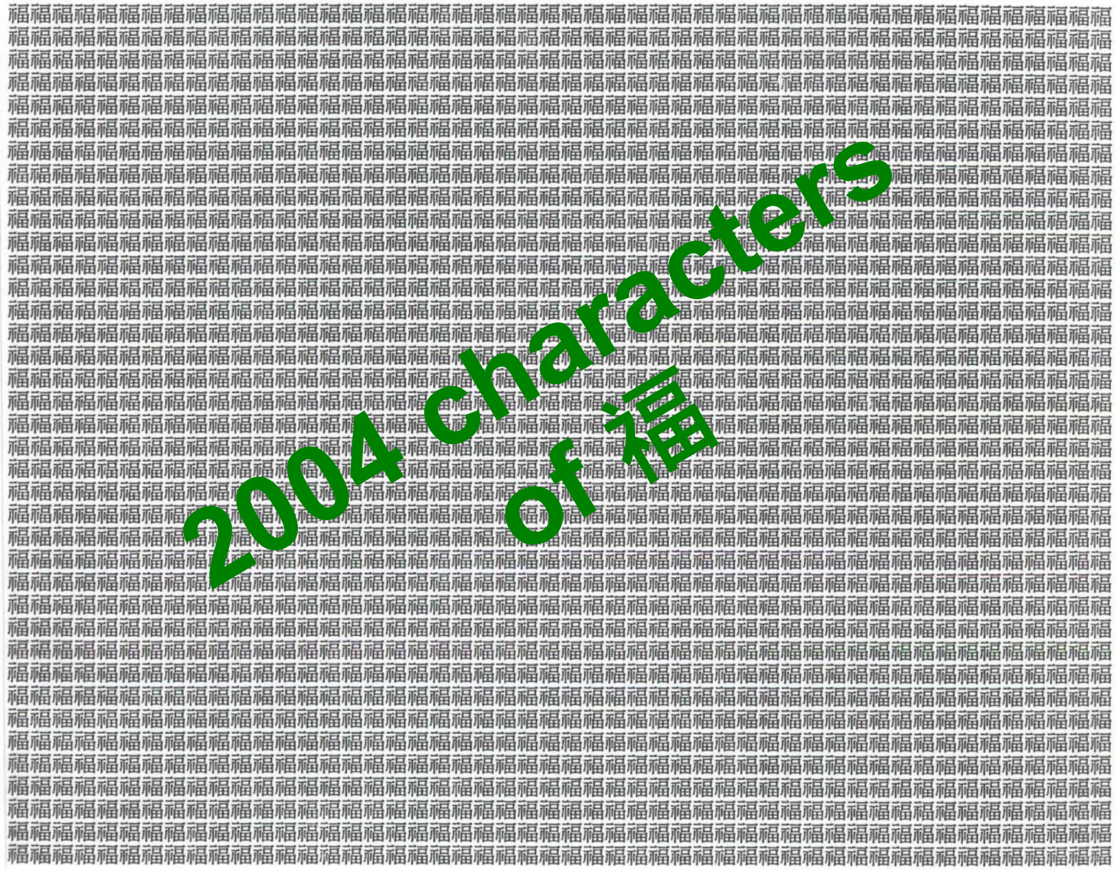
**Algebra is generous, she often gives more than is asked of her [L'algébre est généreuse, elle donne souvent plus qu'on lui demande].**

**Jean le Rond d'Alembert (1717-1783)**

[quoted by Edward Kasner in : The present problems of geometry, *Bull. Amer. Math. Soc.* XI (1905), p.285]

On Sat, 17 Jan 2004, Wong Ngai Ying wrote:

這裡共有2004個福字代表著今年很幸福，希望您轉寄給你其他的好朋友， 祝你好運 !!




Dear NY,

Thank you very much for the good wish for happiness. However, I do not think there are 2004 characters there (why?)

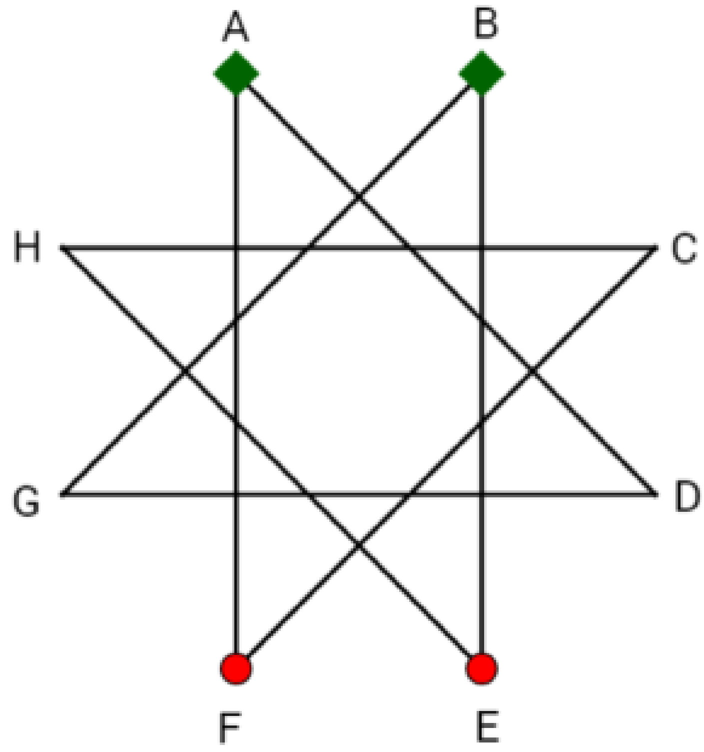
MK

January 17, 2004.

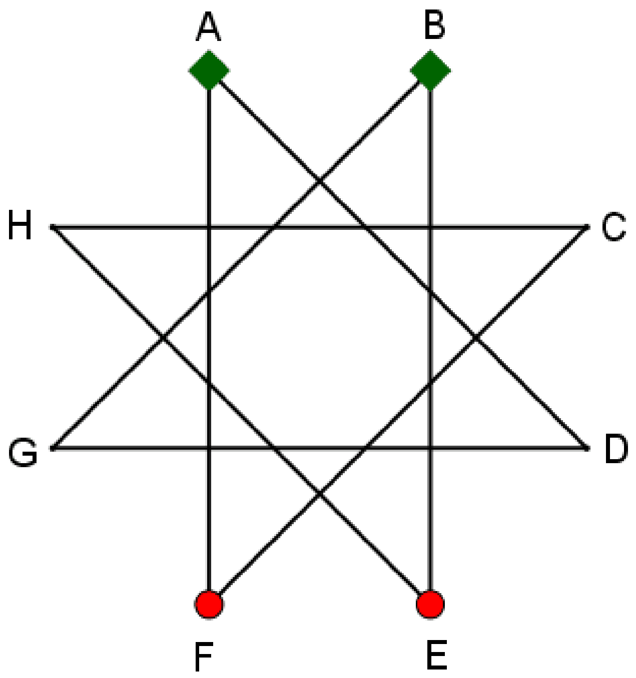
**Without counting how do I know there cannot be 2004 characters there?**

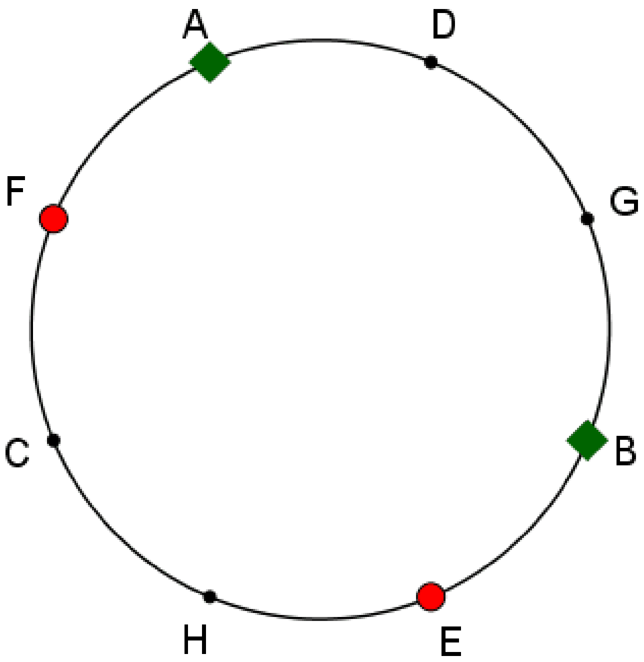
**Rule:** Each piece can be moved along a straight line to an unoccupied position.

**Aim:** To interchange the postions of the coloured pieces.




https://ggbm.at/7182500



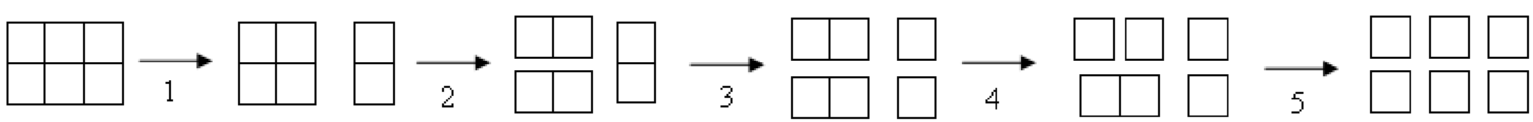

# Isomorphic Games


(a) How many football matches need to be played in a knock-out competition with 10 teams to produce the champion (where no draw is allowed) ?

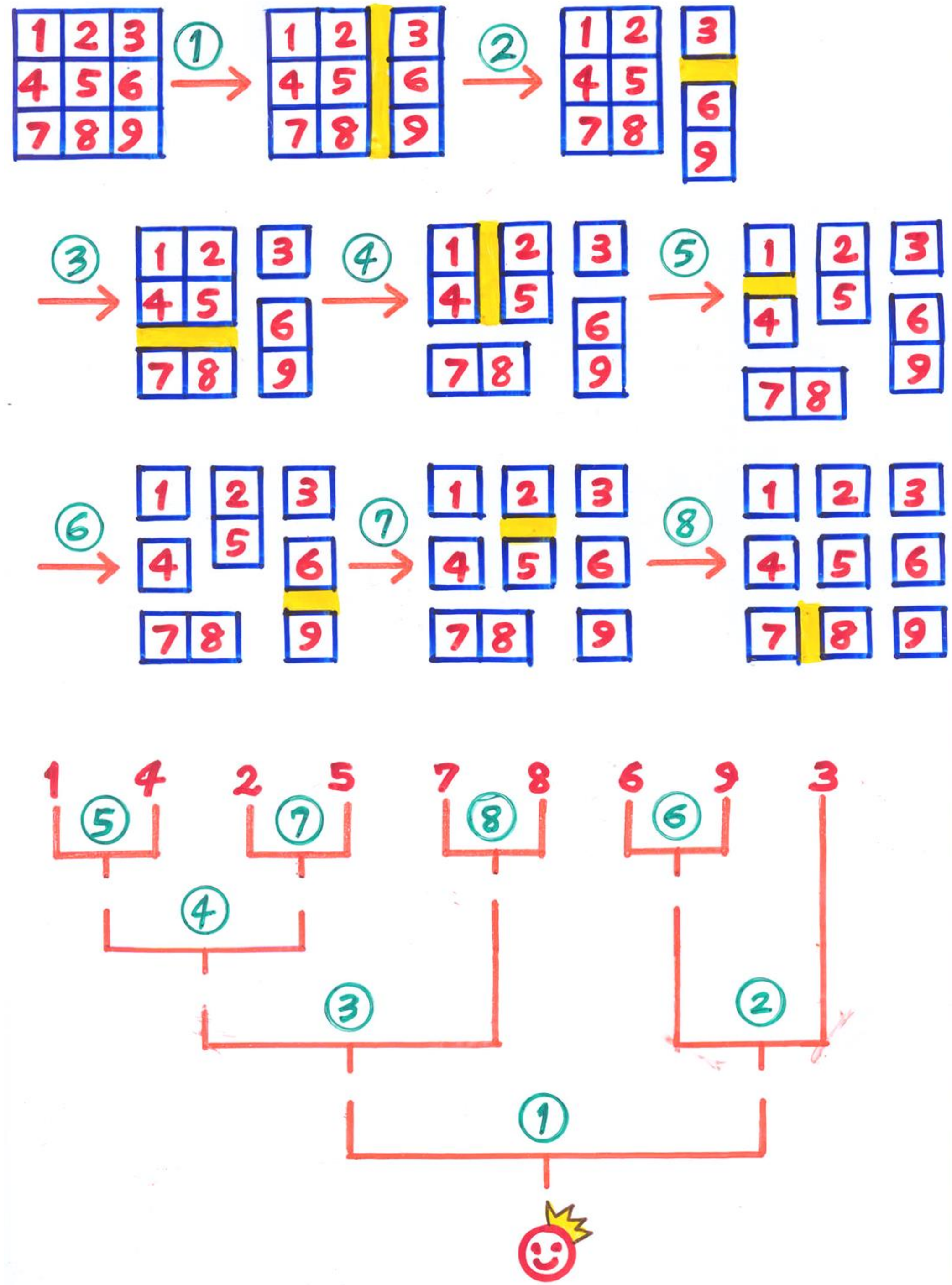
What is the answer in the general case with $N$ teams ?


(b) How many times do you need to break up a rectangular pack of chocolate bar with $m$ rows and $n$ columns to get all $mn$ separate pieces?

e.g.



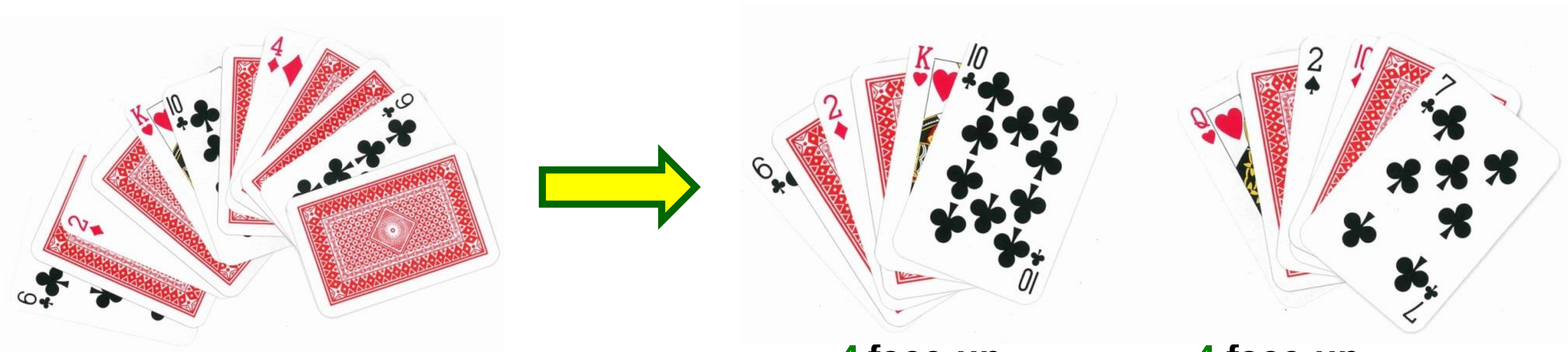

**Are the two problems in (a) and (b) essentially the same? Explain your answer.**



# Isomorphic Problems

**Shuffle a pack of cards, with half of them face-up and half of them face-down.**

**Divide the pack in half with the same number of cards face-up and face-down in each half.**



**4 face-up**
**2 face-down**

**4 face-up**
**2 face-down**

# A magic card trick invented by Bob Hummer

[See: Chapter 1, Martin Gardner, *Mathematics, Magic and Mystery* (1956)]

# In general ...

***N*** **face-up**

***N*** **face-down**

***a*** **face-up**

***N - a*** **face-down**

***N - a*** **face-up**

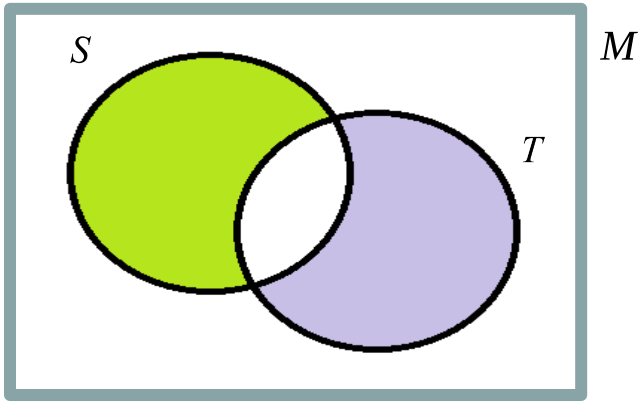
***a*** **face-down**

***a*** **face-up**

***N - a*** **face down**

***a*** **face-up**

***N - a*** **face-down**




If $m(S) = m(T)$,
then $m(S \setminus T) = m(T \setminus S)$.

$M$ = the set of all 52 cards;

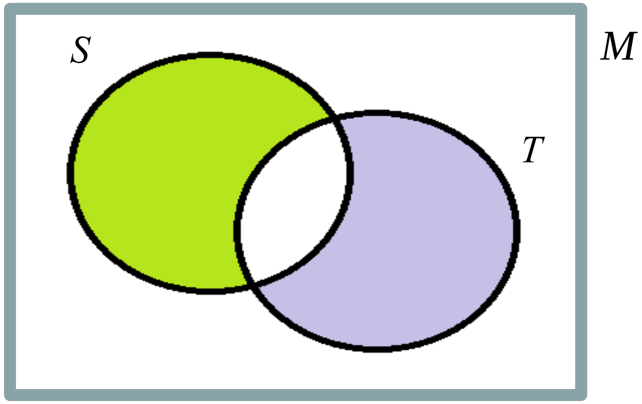
$S$ = the set of 26 face-up cards;

$T$ = the set of 26 cards in one pile $(A)$.

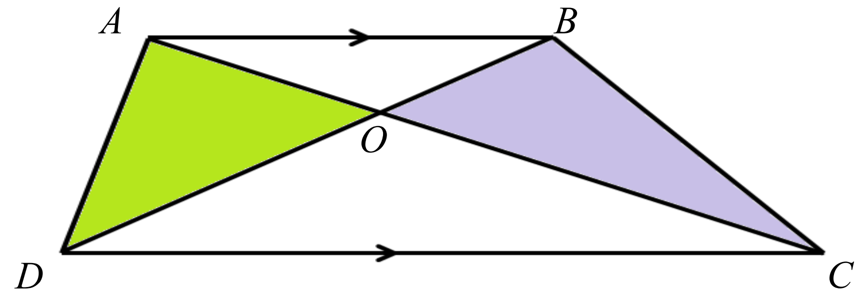
Then $(S \setminus T)$ = the set of face-up cards in the other pile $(M - A)$;

$(T \setminus S)$ = the set of face-down cards in the pile $A$.

**By reversing one of the two piles, the number of face-up cards in each will be the same.**




If $m(S) = m(T)$,
then $m(S\setminus T) = m(T\setminus S)$.

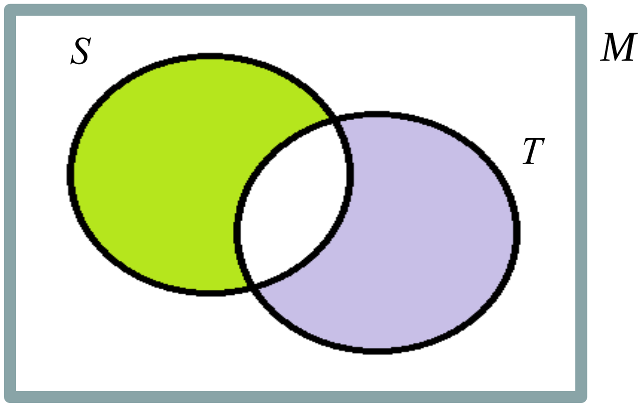



$ABCD$ is a trapezium with $AB$ parallel to $DC$.
$AC$ and $BD$ intersect at $O$.
Then $\triangle AOD$ and $\triangle BOC$ have the same area.

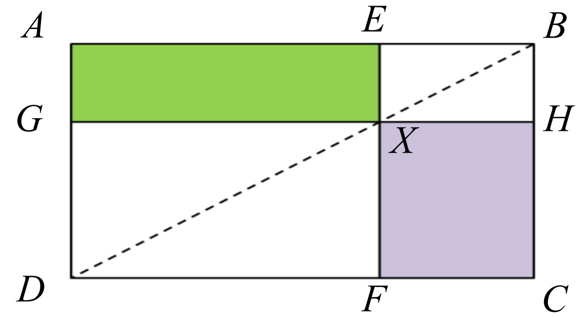
Let $S = \triangle ACD$;

$$T = \triangle BCD.$$

Since $\triangle ACD$ and $\triangle BCD$ have the same area, it follows that $\triangle AOD$ and $\triangle BOC$ have the same area.

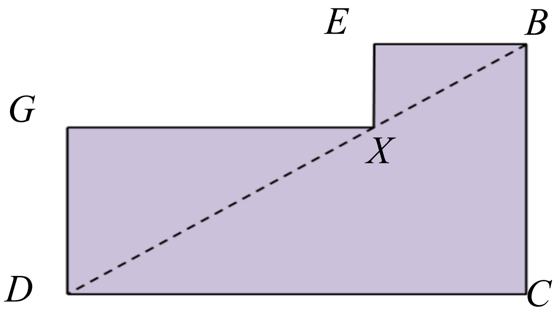


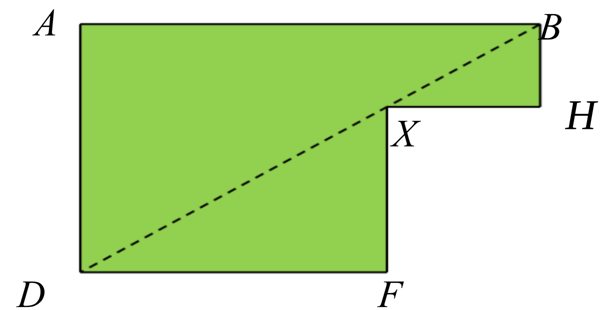

If $m(S) = m(T)$,
then $m(S \setminus T) = m(T \setminus S)$.




$ABCD$ is a rectangle, and $X$ is a point on the diagonal $BD$.
$EXF$ is a parallel to $AD$; $GXH$ is parallel to $AB$.

Then the rectangles $AEXG$ and $XHCF$ have equal area.




Let $S = CDGXEB$

Let $T = ABHXFD$

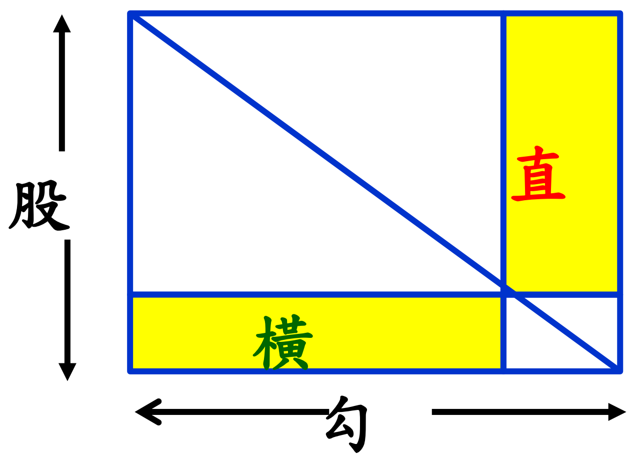
Since $CDGXEB$ and $ABHXFD$ have the same area, it follows that $AEXG$ and $XHCF$ have the same area.

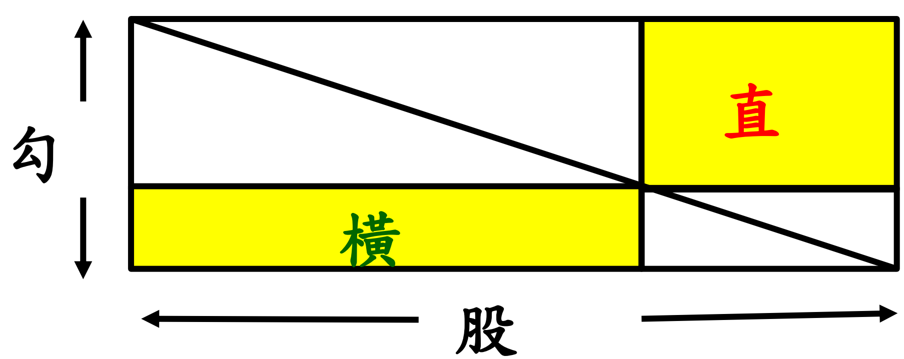
勾(股)中容橫。股(勾)中容直。
二積皆同。古人以題易名。
若非釋名。則無以知其源。

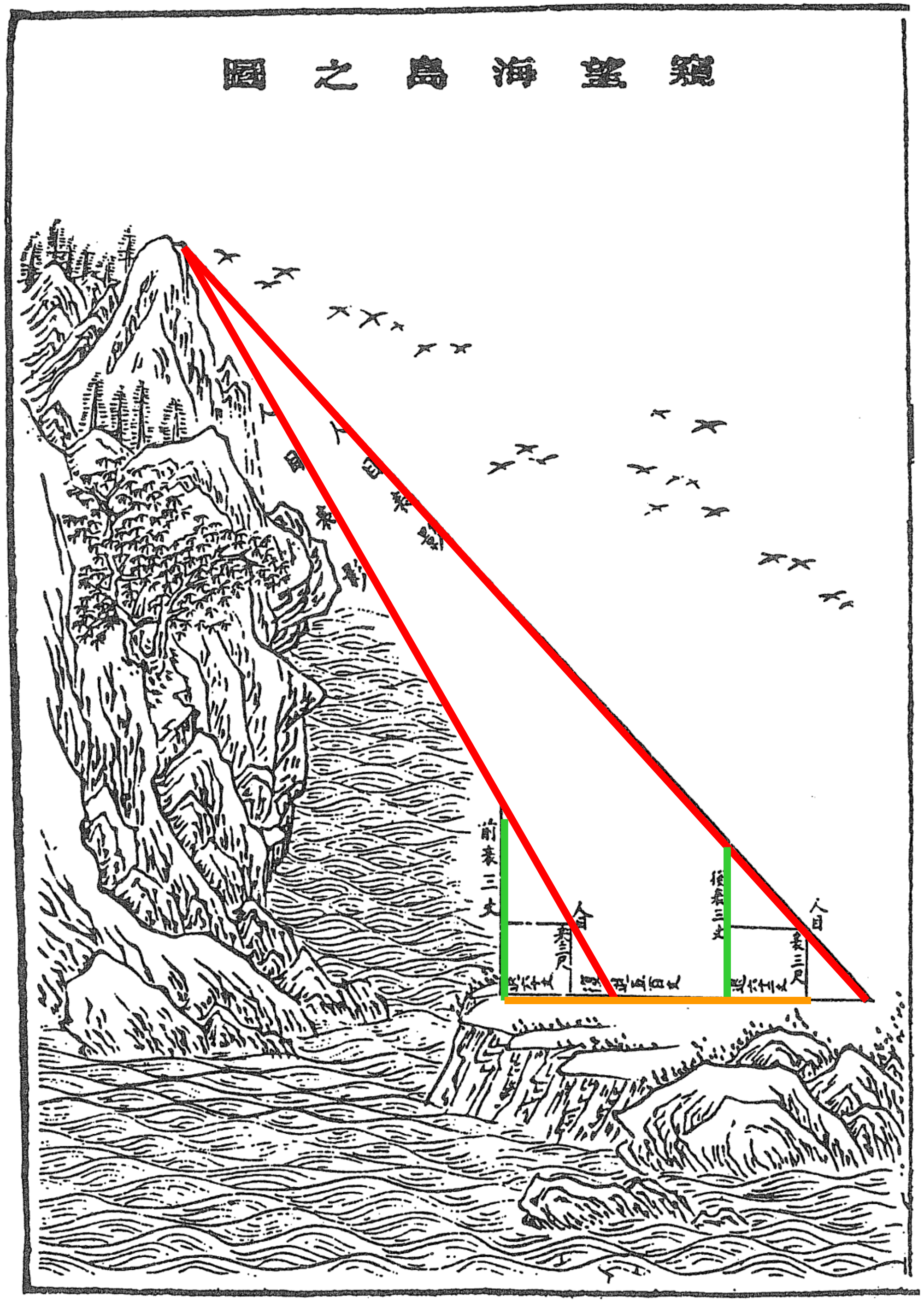
(The horizontal rectangle formed by part of the base and the vertical rectangle formed by part of the perpendicular are equal in area. Men of the past changed the names of their methods from problem to problem …)

Compare with Proposition 43 of Book I of Euclid's *Elements.*

楊輝，《續古摘奇算法(卷下)》
YANG Hui, *Continuation of Ancient Mathematical Methods for Elucidating the Strange* [*Properties of Numbers*] (Chapter II) (1275)






**Method of Double-Difference**

**of LIU Hui [劉徽]**

**in *Haidao Suanjing***

**[海島算經 Sea Island Mathematical Manual]**

**(3rd century)**

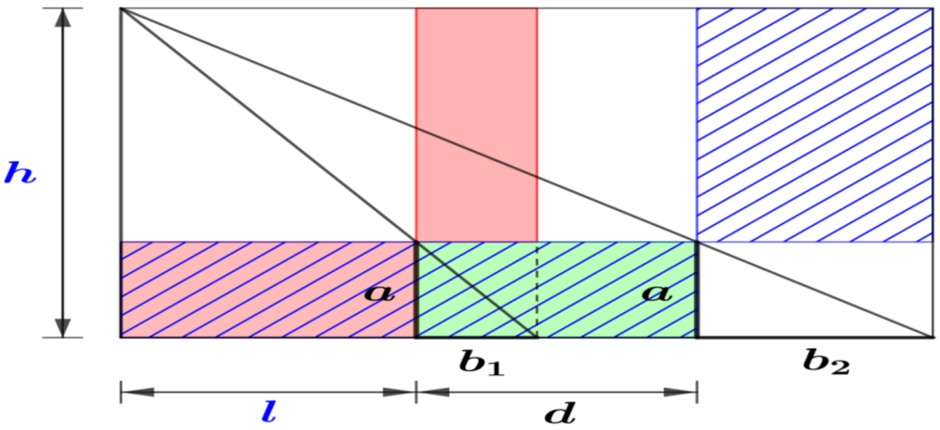
**as illustrated in**

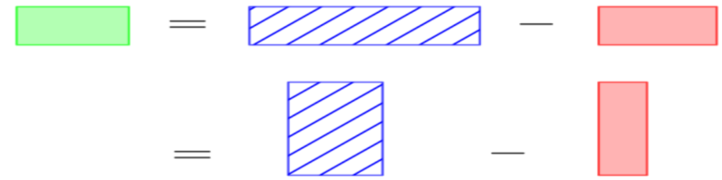
***Gujin Tushu Jicheng***

**[古今圖書集成 Complete Collection of Pictures and Writings of Ancient and Modern Times]**

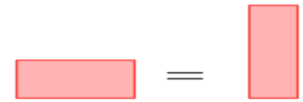
**(1726)**

Given $a$, $d$, $b_1$ and $b_2$, how can we express $h$ and $l$ in terms of $a$, $d$, $b_1$ and $b_2$ ?



$$\therefore \quad ad = (h-a)b_2 - (h-a)b_1$$
$$= (h-a)(b_2-b_1)$$
$$h = \frac{ad}{b_2-b_1} + a$$



$$\therefore \quad la = b_1(h-a) = \frac{b_1 ad}{b_2-b_1}$$
$$l = \frac{b_1 d}{b_2-b_1}$$

http://ggbtu.be/m2812113

**Explanation by YANG Hui on the Method of Double-Difference of LIU Hui (1275)**

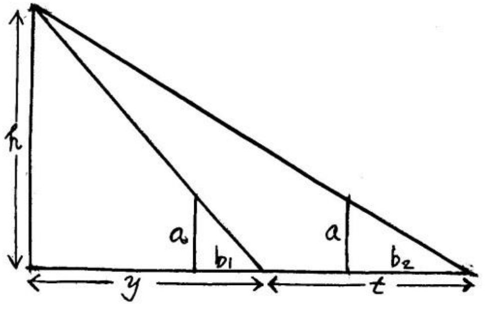
# Aryabhata I 阿耶波多

(c.476-550)


**Aryabhatiya 《阿耶波多曆數表》**

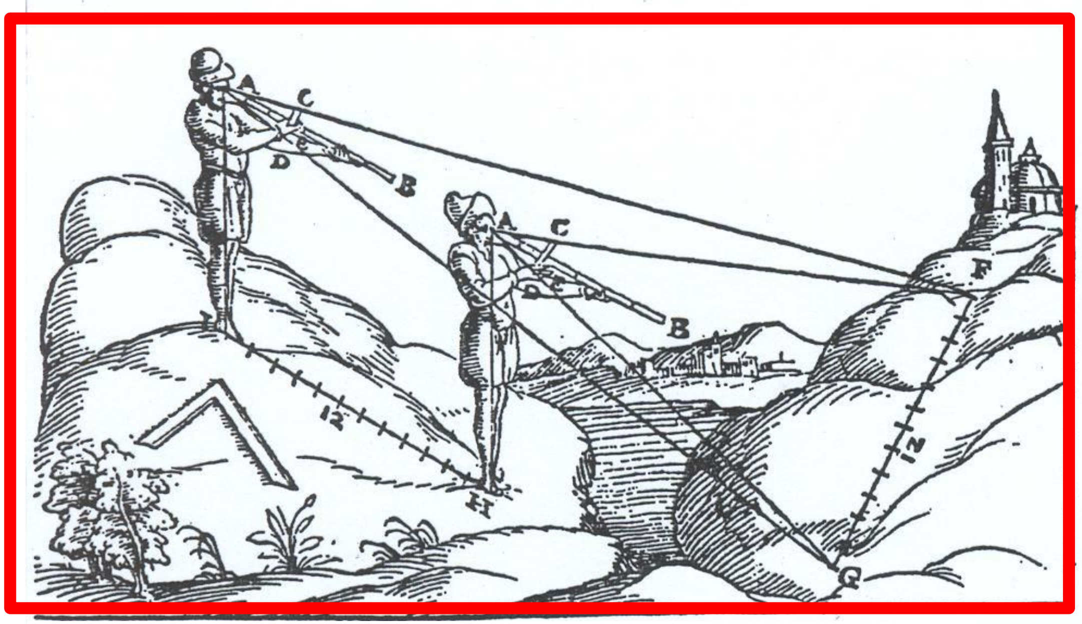
## Book II, Stanza 16

The distance between the ends of the two shadows multiplied by the length of the first shadow and divided by the difference in length of the two shadows gives the *koṭī* . The *koṭī* multiplied by the length of the gnomon and divided by the length of the (first) shadow gives the length of the *bhujā* .




$$y = \frac{tb_1}{b_2 - b_1}$$

$$h = \frac{ya}{b_1} \quad (= \frac{ta}{b_2 - b_1})$$




Orence Fine, *De re & praxi geometrica* (1556)

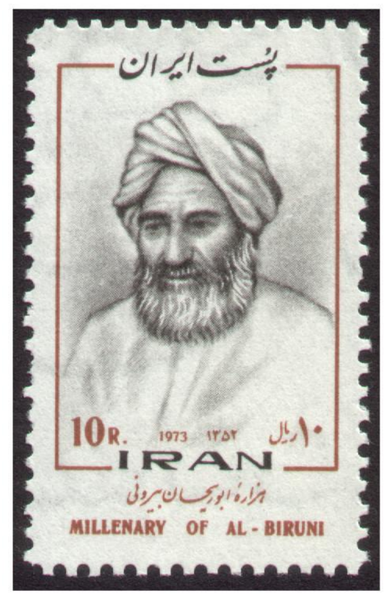

A Jacob's staff, from John Sellers' *Practical Navigation* (1672)

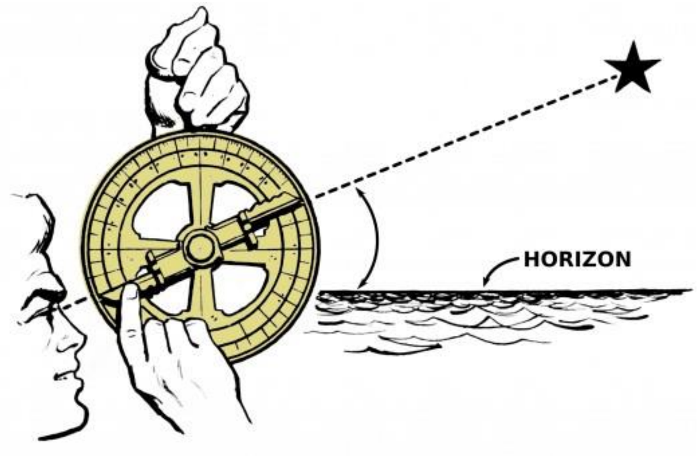
John Sellers, *Practical Navigation* (1672)

**The invention of the cross-staff (or Jacob's staff) has been credited to Levi ben Gerson (1288-1344).**

# Abū Rayhān Muhammad ibn Ahmad Al-Bīrūnī [usually known As Al-Biruni] (973-1048)

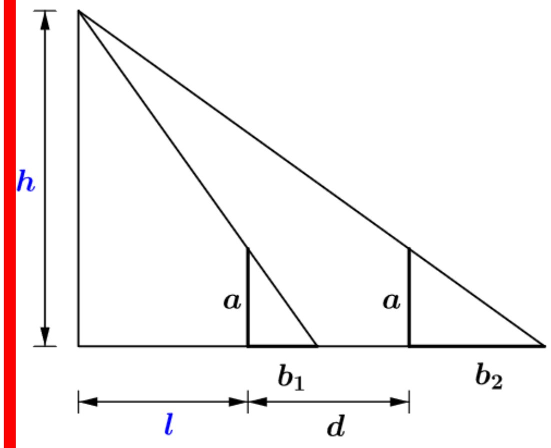
**Statue of Al-Biruni in Laleh Park, Tehran, Iran.**

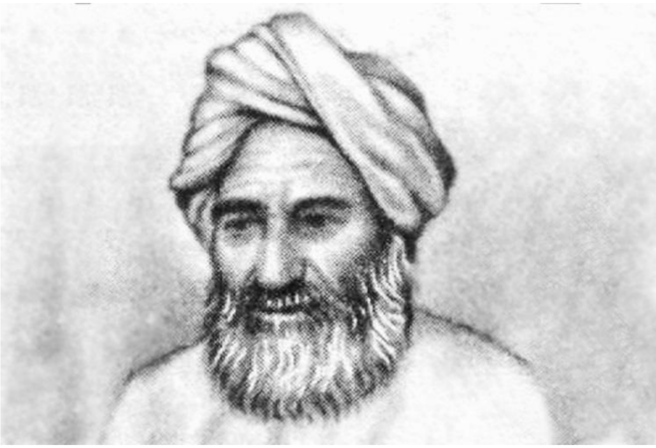




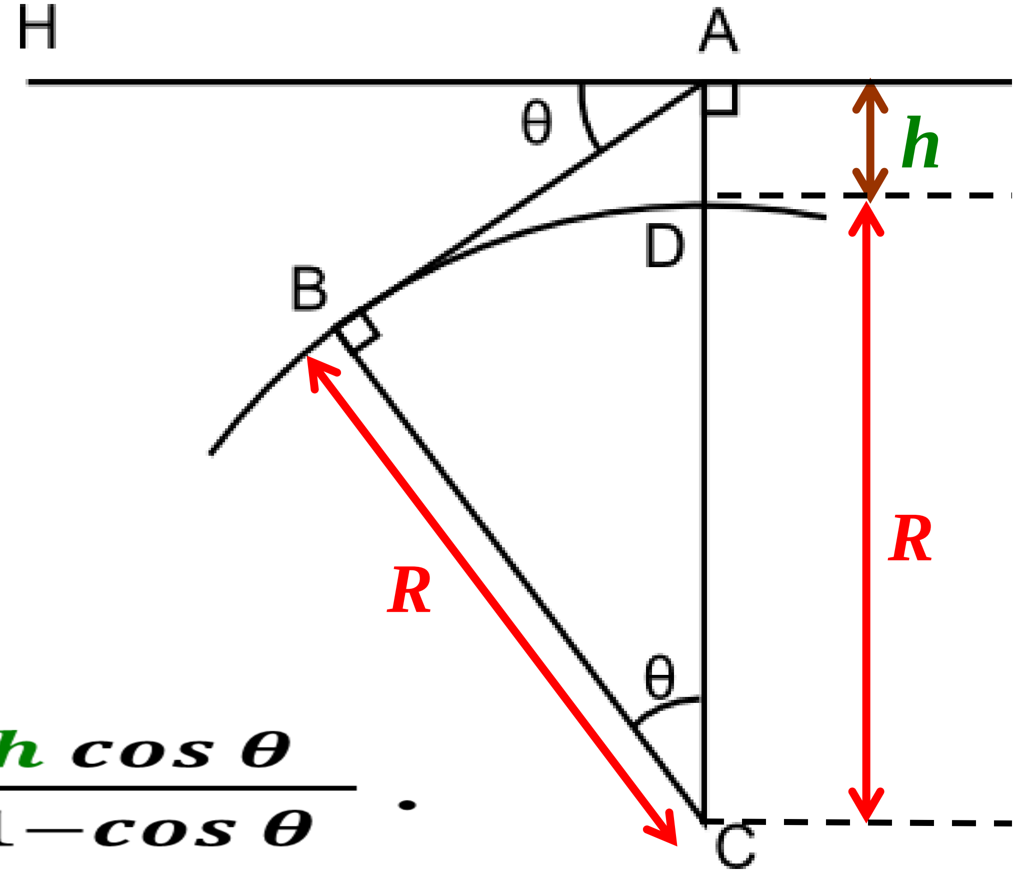

# Measuring (inaccessible) height with an astrolabe

Given $a$, $d$, $b_1$ and $b_2$, how can we express $h$ and $l$ in terms of $a$, $d$, $b_1$ and $b_2$ ?





Al-Biruni (973-1048)



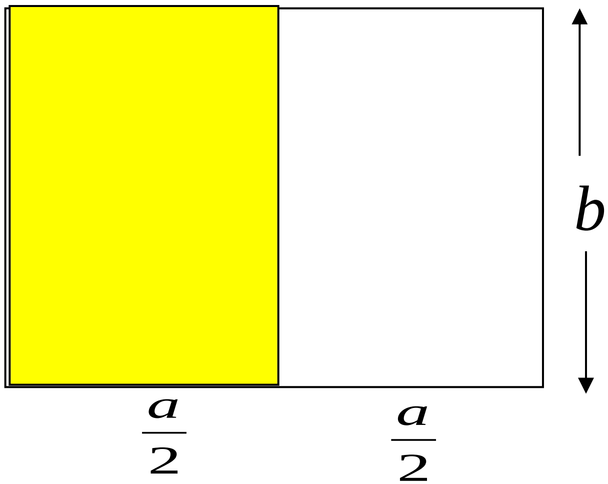

$$\frac{R}{h+R} = \cos\theta\ ,$$

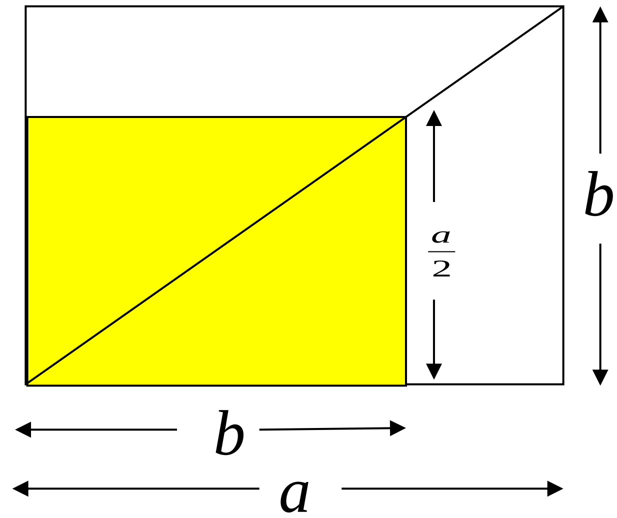
$$\text{Hence,}\quad R = \frac{h\cos\theta}{1-\cos\theta}\ .$$

## Measurement of the size of the earth by Al-Biruni :

First measure the height $h$ of a mountain. From the top of the mountain measure the angle of depression $\theta$ of the horizon. Then compute the radius of the earth $R$ by using trigonometry.

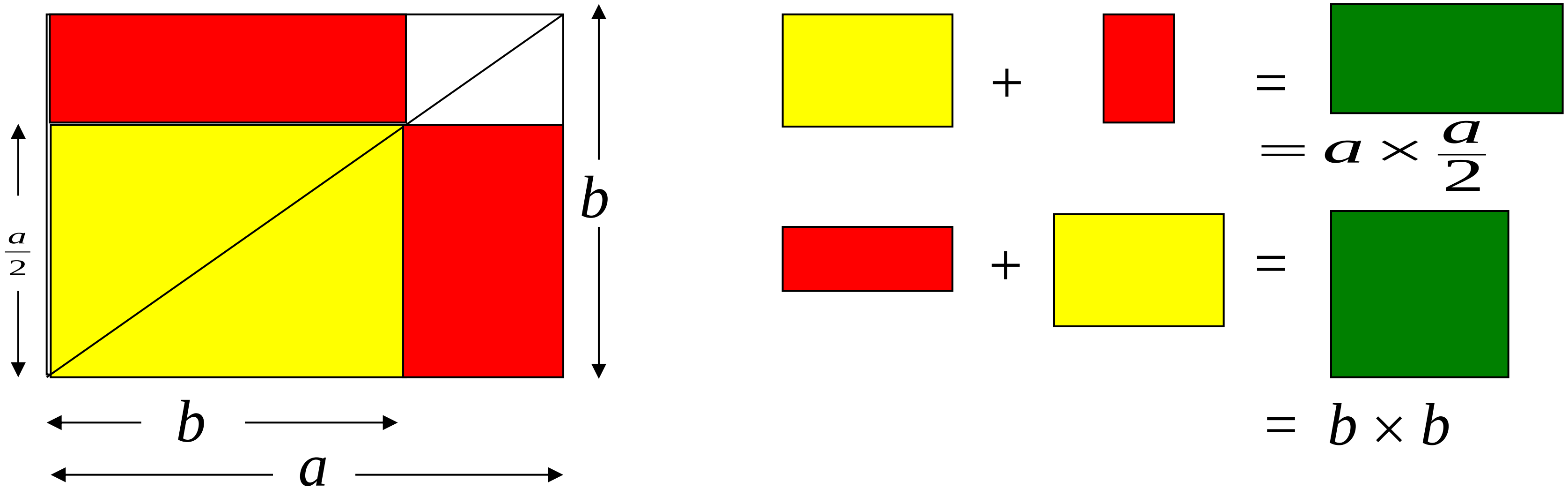
# Size of A3 and A4 paper




$a = ?\ ,\ b = ?\ .$

**Factor of enlargement and reduction in a photocopier**




$$a \times \frac{a}{2} = b \times b$$

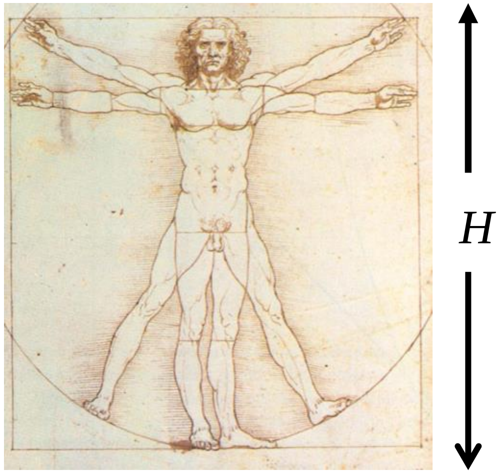
$$a \times a = 2 \times b \times b$$

$$a = \sqrt{2} \times b$$

**Magnifying factor = (√2) × 100 % ≈ 141 %**

**Shrinking factor = (1/√2) × 100 % ≈ 71 %**

**A man weighing 50*Kg* can normally lift up 30*Kg* . How much can a man weighing 100*Kg* normally lift up?**




**Weight (*W*) is proportional to the cube of height.**

**Weight capable of lifting up (*F*) is proportional to the cross-sectional area of the muscle, hence proportional to the square of height (*H*).**

$$\frac{W_1}{W_2}=\frac{H_1^{\,3}}{H_2^{\,3}},\quad \frac{F_1}{F_2}=\frac{H_1^{\,2}}{H_2^{\,2}},$$

Therefore $\boxed{\dfrac{W_1^{\,2}}{W_2^{\,2}}=\dfrac{F_1^{\,3}}{F_2^{\,3}}}$ .

If $W_1=2W_2$, then $\dfrac{F_1^{\,3}}{F_2^{\,3}}=4$, $\dfrac{F_1}{F_2}=\sqrt[3]{4}=1.5874\ldots$

Since $F_2$ is $30Kg$, $F_1$ is computed to be $47.62\ldots Kg$ .

**An ant normally measures 0.005m. It can carry a burden that is 5 times its own weight. If a giant ant were as big as a man (say of height 1.75m), how much times of its own weight would it be able to carry?**

$$H_2 \qquad\qquad H_1 = \frac{1.75}{0.005} H_2 = 350 H_2$$

$$W_1 = 350 \times 350 \times 350 \times W_2, \quad F_1 = 350 \times 350 \times F_2,$$

It is known that $\dfrac{F_2}{W_2} = 5$,

therefore $\dfrac{F_1}{W_1} = \dfrac{350 \times 350 \times F_2}{350 \times 350 \times 350 \times W_2}$

$$= \frac{1}{350} \times \frac{F_2}{W_2} = \frac{1}{70},$$

**The giant ant can only carry 1/70 of its own weight. It can hardly stand on its own feet!**

# Allocation problems

occur frequently in everyday life, for example, sharing the expense of a meal, allocating work among a group of people, sharing a prize or goodies, apportioning seats of legislators for different districts, … .
Is it easy? Is it difficult?

**Does knowledge in division alone suffice?**

# The Lion's Share

The Lion went once a-hunting along with the Fox, the Jackal, and the Wolf. They hunted and they hunted till at last they surprised a Stag, and soon took its life. Then came the question how the spoil should be divided. “Quarter me this Stag,” roared the Lion; so the other animals skinned it and cut it into four parts. Then the Lion took his stand in front of the carcass and pronounced judgment: “The first quarter is for me in my capacity as King of Beasts; the second is mine as arbiter; another share comes to me for my part in the chase; and as for the fourth quarter, well, as for that, I should like to see which of you will dare to lay a paw upon it.”

“Humph,” grumbled the Fox as he walked away with his tail between his legs; but he spoke in a low growl.

"You may share the labours of the great,

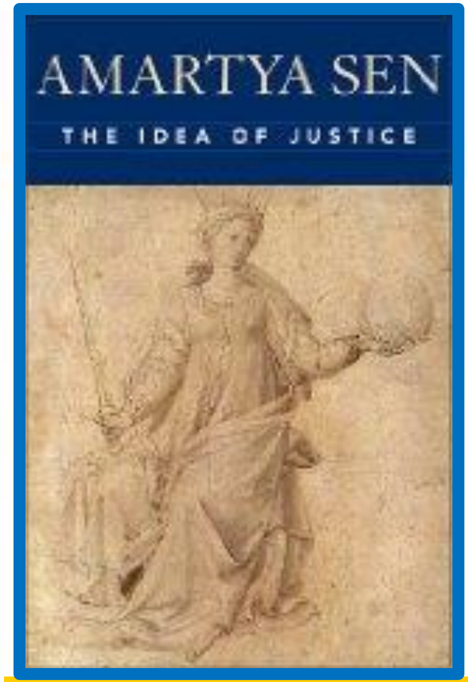
but you will not share the spoil."

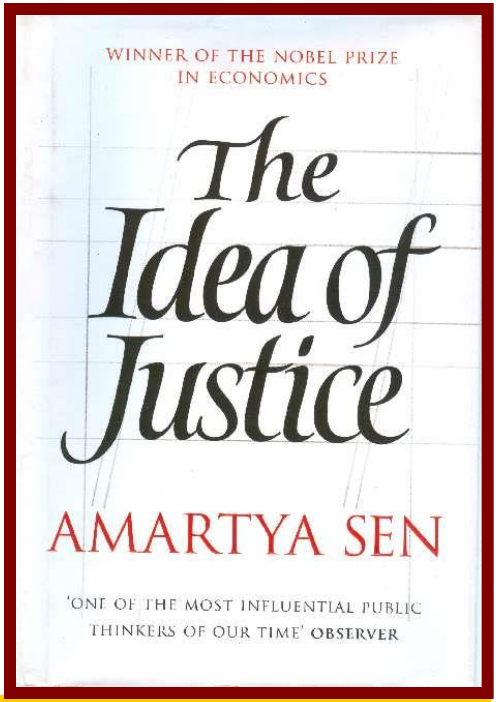
**Æsop’s Fables 伊索寓言**

**(c. 600 B.C.)**

# Principles of Allocation
# 分配原則

- 公開 Open
- 公平 Fair
- 公正 Just




# *The Idea of Justice*

## Amartya Sen (2009)

**Amartya Kumar Sen (1933- )**
**Nobel Memorial Prize in Economic Sciences, 1998**

**A, B, C shared a taxi to go home. A's fare was $20, B's fare was $60 and C's fare was $80. A and B got off before C, and C paid $80 when he reached home. Putting aside friendly generosity how should the expense be shared?**

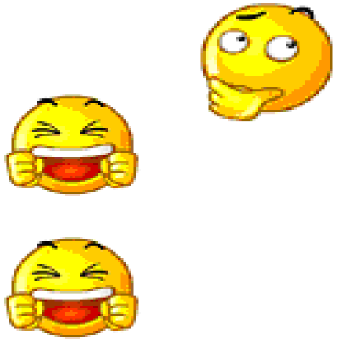
# Plan 1 :

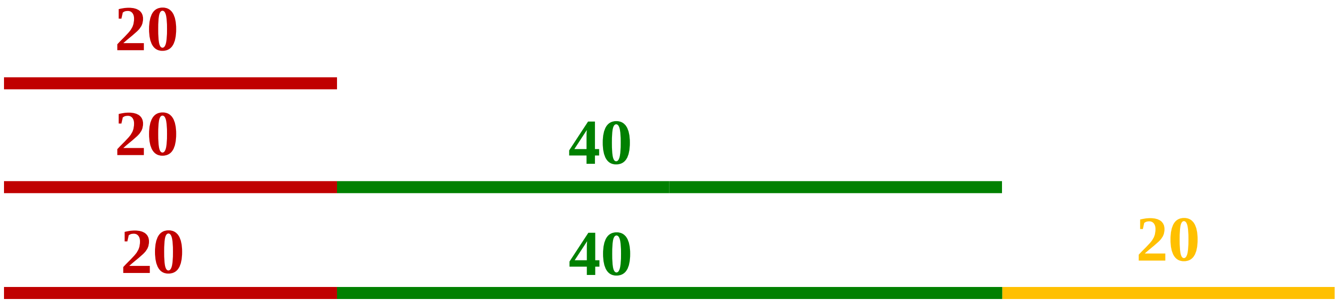
## Shared the expense equally.

**A paid $26.70 ,**

**B paid $26.70 ,**

**C paid $26.70 .**





A, B, C shared a taxi to go home. A's fare was $20, B's fare was $60 and C's fare was $80. A and B got off before C, and C paid $80 when he reached home. Putting aside friendly generosity **how should the expense be shared?**

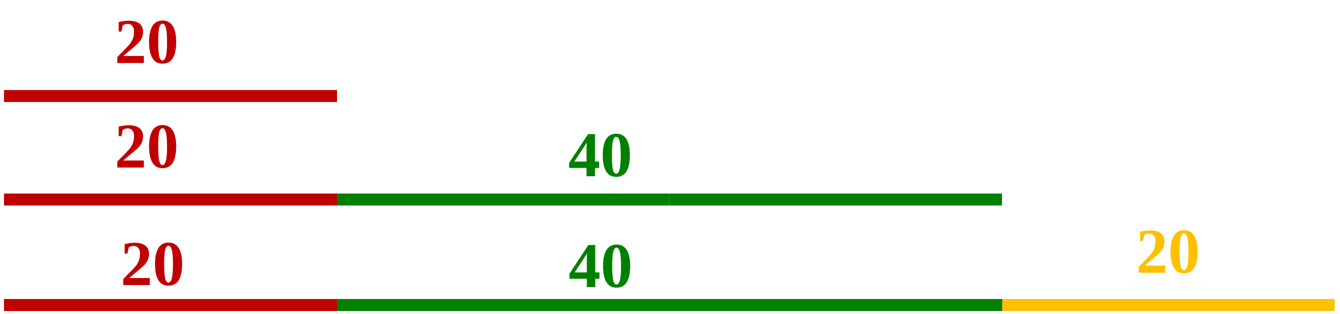
## Plan 2 :

**By sharing a taxi they saved $160 - $80 = $80, so this sum should be divided equally and be deducted from what each should have paid if taking a taxi alone.**

**A paid - $6.60 (i.e. gained $6.60!) ,**

**B paid $33.40 ,**

**C paid $53.40 .**



**A, B, C shared a taxi to go home. A's fare was $20, B's fare was $60 and C's fare was $80. A and B got off before C, and C paid $80 when he reached home. Putting aside friendly generosity how should the expense be shared?**

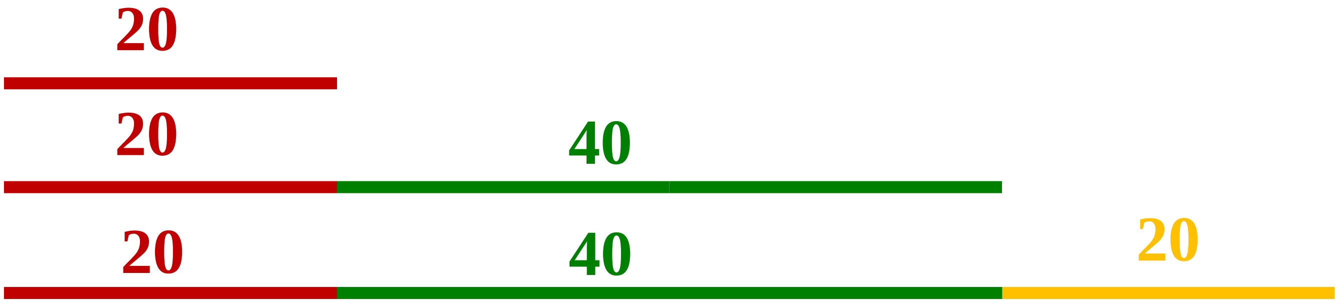
# Plan 3 :

**Shared by proportion 20:60:80.**

**A paid $80 × 1/8 = $10 ,**

**B paid $80 × 3/8 = $30 ,**

**C paid $80 × 4/8 = $40 .**



**A, B, C shared a taxi to go home. A's fare was \$20, B's fare was \$60 and C's fare was \$80. A and B got off before C, and C paid \$80 when he reached home. Putting aside friendly generosity how should the expense be shared?**

## Plan 4 :

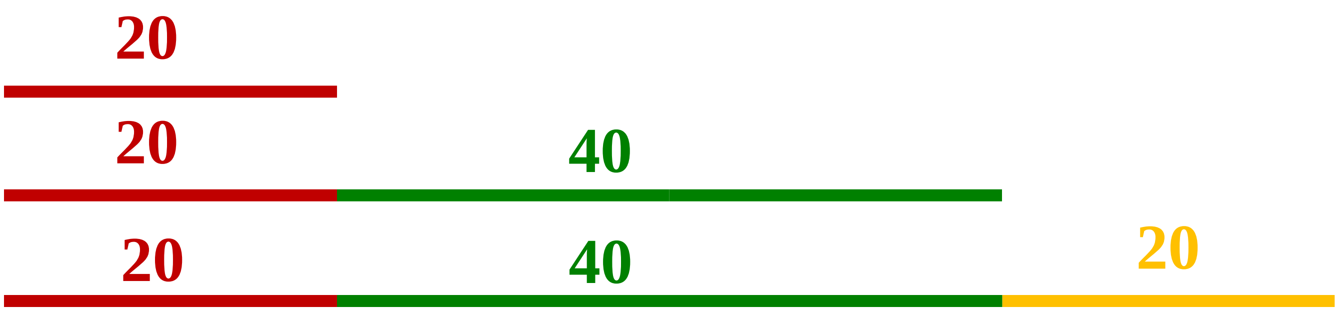
A paid $\$20 \times 1/3 = \$6.70$ ,

B paid $\$20 \times 1/3 + \$40 \times 1/2$

$= \$26.70$ ,

C paid $\$20 \times 1/3 + \$40 \times 1/2 + \$20$

$= \$46.70$ .



A, B, C shared a taxi to go home. A's fare was $20, B's fare was $60 and C's fare was $80. A and B got off before C, and C paid $80 when he reached home. Putting aside friendly generosity **how should the expense be shared?**

## Plan 4 :

A paid $\$20 \times 1/3 = \$6.70$ ,

B paid $\$20 \times 1/3 + \$40 \times 1/2$

$= \$26.70$ ,

C paid $\$20 \times 1/3 + \$40 \times 1/2 + \$20$

$= \$46.70$ .

# How to balance and assess the pros and cons of different plans?

A, B, C shared a taxi to go home. A's fare was \$20, B's fare was \$60 and C's fare was \$80. A and B got off before C, and C paid \$80 when he reached home. Putting aside friendly generosity how should the expense be shared?

## Plan 4 :

A paid $\$20 \times 1/3 = \$6.70$ ,

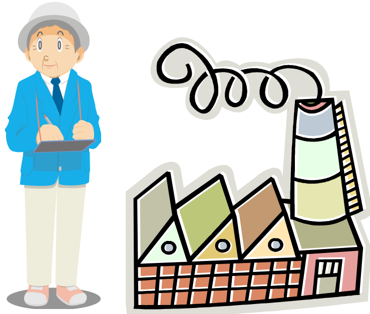
B paid $\$20 \times 1/3 + \$40 \times 1/2$

$= \$26.70$ ,

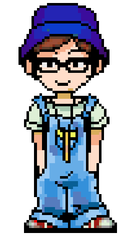
C paid $\$20 \times 1/3 + \$40 \times 1/2 + \$20$

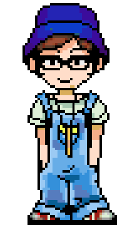
$= \$46.70$ .

**Is there any other alternative plan?**
**On what basis is the sharing of expense carried out?**


A


B


C


D

A + B profit of $100 m.

A + B + C profit of $500 m

A + B + D profit of $500 m

A + B + C + D profit of $2000 m

No A , there is no factory.

No B , the factory cannot operate.

**How should the profit of $2000m be shared fairly?**

*A* : I own the factory. If there is no factory, there is no job and no profit. Hence I should take \$1000m, *B* takes \$500m, and *C* , *D* each takes \$250m.

*B* : That is not fair! The factory cannot operate without me. I should get at least as much as *A*. Split the \$1500m between us (*A* and me), each getting \$750m. *C* and *D* each gets \$250m.

*C* : That is not fair! I earn for *A* and *B* an extra \$400m. I should take \$400m, and so should *D*. *A* and *B* can split the remaining sum, so each gets \$600m.

*D* : No, we earn for them an extra \$1900m. Each of us (*C* and me) should take \$950m. Let *A* and *B* split the remaining sum, so each gets \$50m.

*B* : Come on, you do not have your job if there is no factory. You cannot operate without me either. We should at least share that extra \$1900m equally among ourselves, each getting \$475m. Then *A* and I each gets an extra \$50m, making \$525m.

*C* : Why don't we simply divide \$2000m equally among ourselves, each getting \$500m?

*A* : That is not fair, because I own the factory!

**A**: I own the factory. If there is no factory, there is no job and no profit. Hence I should take $1000m, *B* takes $5 nd *C* , *D* each takes $250m.

**B**: That is not fair! The factory canno I should get at least as much as *A* us (*A* and me), each getting $75

**C**: That is not fair! Le take $400m, and sum, so ea

**D**: N m sum, so

**B**: ere is no factory. You c We should at least share that ex ourselves, each getting $475m. gets an extra $50m, making $525m.

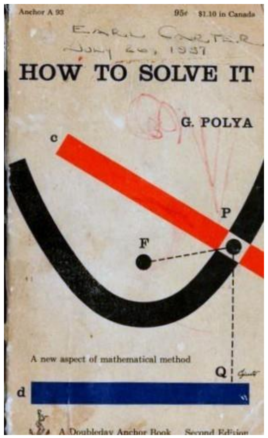
**C**: Why ply divide $2000m equally among ourselves, getting $500m?

**A**: That is not fair, because I own the factory!

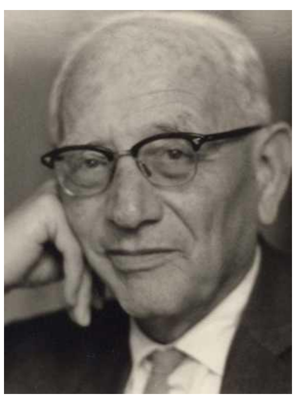
**More about this problem in the next lecture**

**— the solution leads to a Nobel Memorial Prize in Economic Sciences of 2012!**

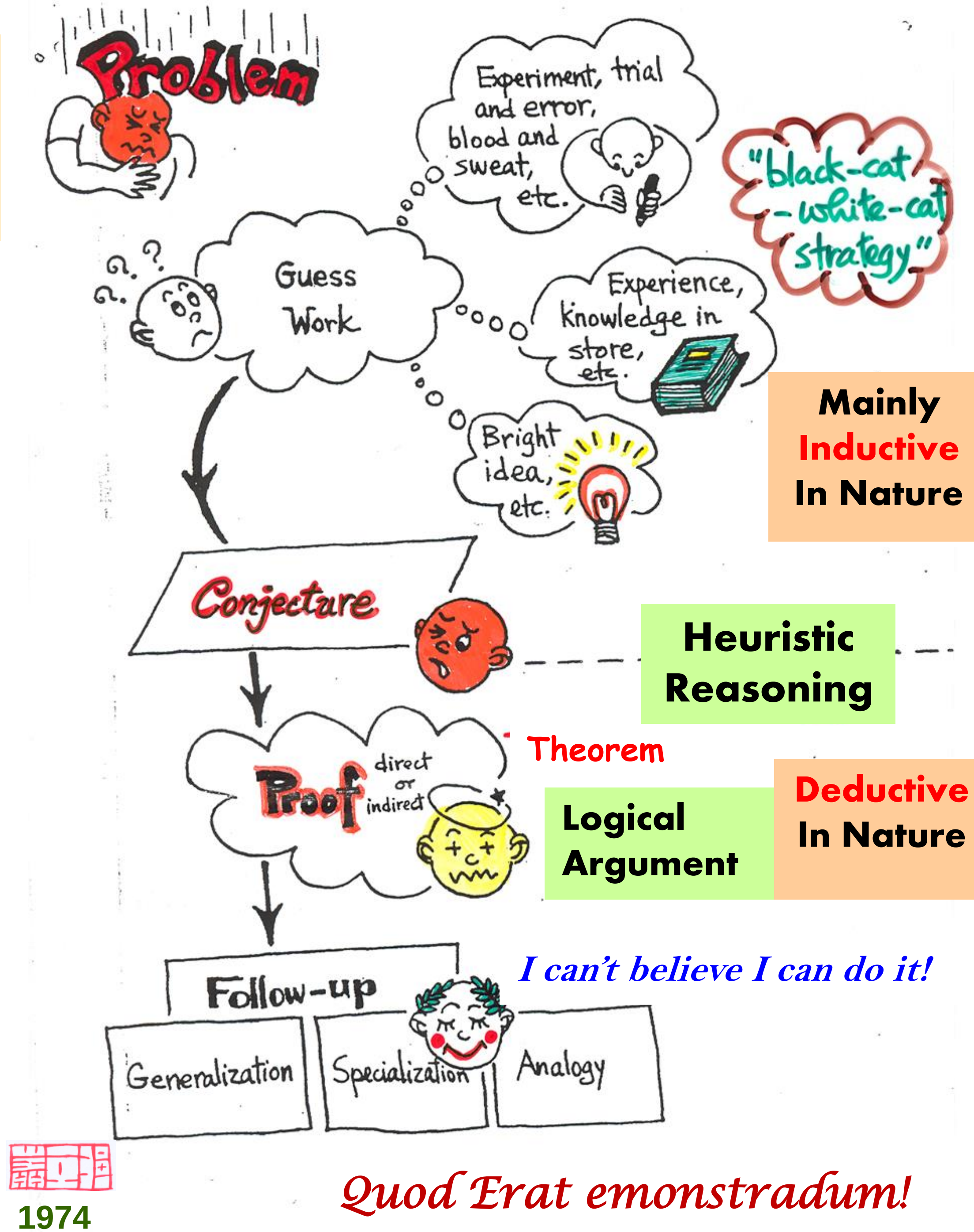
# Steps in Problem Solving à la Polya




**George Pólya**
***How To Solve It***
**1957 Edition**
**(first published in 1945)**



George Pólya
(1887-1985)




*Quod Erat emonstradum!*

# Question :

Try to partition $N$ consecutive numbers

$$1, 2, 3, \ldots, N$$

into two subsets with the sum of the numbers in one subset equal to that of the numbers in the other subset.

For precisely which $N$ is this possible?

**Can we partition 1, 2 this way?**

**Can we partition 1, 2, 3 this way?**

**Can we partition 1, 2, 3, 4 this way?**

**Can we partition 1, 2, 3, 4, 5 this way?**

# Question :

$6 = 1 + 2 + 3$ is the sum of some consecutive positive integers, and so is
$7 = 3 + 4$.
Try to find all $N$ that can be written as a sum of (at least two) consecutive positive integers.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | | 3 | 6 | 10 | 15 | 21 | 28 | 36 |
| 2 | | | 5 | 9 | 14 | 20 | 27 | 35 |
| 3 | | | | 7 | 12 | 18 | 25 | 33 |
| 4 | | | | | 9 | 15 | 22 | 30 |
| 5 | | | | | | 11 | 18 | 26 |
| 6 | | | | | | | 13 | 21 |
| 7 | | | | | | | | 15 |

Each row is the sum of two, three, four, five, …… consecutive positive integers starting with the number at the head of the column.

**Do you notice what numbers are missing?**

# Question :

N persons went to meddle with N closed lockers in a row. The first person opens all lockers. The second person closed every second locker starting with the $2^{nd}$ one. The third person changed the state (opened or closed) of every third locker starting with the $3^{rd}$ one, and so on until the $N^{th}$ person changed the state of the $N^{th}$ locker.

**At the end which lockers were left open?**