

The Power and “*Way*” of Mathematics

Part I

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Most people, even if they do not study mathematics, will recognize the **utility of mathematics and its **power** in solving problems in various areas and in explaining what is happening in the world around us, not just the physical world but also the biological and the economic-financial world as well.**

What endows mathematics

With *its power and utility*?

What is the “*Way*” of mathematics?

**How did mathematics develop
into a subject as it is today?**

What is the nature of the subject?

**Is mathematics a tool,
or a way of thinking,
or a part of culture?**

**Is a mathematical proof
a kind of ritual observed
by a certain sect
(called mathematicians)?**

**What is a proof for — verification,
or enhancement of understanding, or
training of the mind, or only
for professional conscience?**

**Why are we so certain that
a theorem in mathematics
really holds true?**

How do mathematicians work?

Do they only do **complicated calculations**? Do they just follow **logical deduction**, or do they allow **wild thinking** that may not yet be justified according to logic?

When a mathematician talks about the **beauty** and **elegance** of mathematics, what is meant by that?

Why can mathematics,
abstract and **seemingly**
man-made though it is,
explain and be **applicable** to
so many different phenomena
in the real world?

Do we **discover** the mathematics,
or

do we **invent** the mathematics?

It would take at least a full course to just touch upon such issues, which soon become philosophical. Since one can hardly do justice to such issues in one or two lectures, we will in these two lectures (October 15 and 22) try only to have a glimpse of the power and “*Way*” of mathematics by going through some selected examples. We will work through and discuss these examples in class together, presuming knowledge in school mathematics. In the second lecture we will talk more about proofs in mathematics.

Course YSCN0002

Mathematics: A Cultural Heritage

數學：文化的傳承

This course attempts to elaborate an exhortation of Hermann Weyl through examples gathered from the long history of mathematics, around our daily lives, in other areas of human endeavour and in Nature. Rather than transmitting a body of technical knowledge in mathematics the emphasis is placed on appreciating, contemplating and discussing about the **beauty**, the **utility** and the **“Way”** of mathematics.

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From 1999 to 2009 it was offered as an elective for students in any faculty and department at HKU.

“Anschauung” of Mathematics

數學觀

Sense of

數學之「道」

The “Way” of Mathematics

數學意識

Mathematics

數學之美

Beauty of Mathematics

數學之用

Utility of Mathematics

Knowledge in Mathematics

數學知識

**Nigradoc ot a seerchar ta
na Elnshig nevstriyu, ti sedon't
tamert ni thaw redor het stetler ni
a rowd rae, het loyn torptamin
hingt si hatt rifst nad salt telter ear
ta het girth caple. Het sert anc eb a
lotta sems nad uoy anc tills dear ti
thowitu belmorp. Hist si cabusee ew
od ton dear revey tetler by sitfle tub
het rowd sa a howle.**

Aoccdrnig to a rseerach at an Elingsh uinervtisy, it deosn't mttar in waht oredr the ltteers in a wrod are, the olny iprmoatnt tihng is taht frist and lsat ltteer are at the rghit pclae. The rset can be a toatl mses and you can sitll raed it wouthit porbelm. Tihs is bcuseae we do not raed ervey lteter by itslef but the wrod as a wlohe.

According to a research at an English university, it doesn't matter in what order the letters in a word are, the only important thing is that first and last letter are at the right place. The rest can be a total mess and you can still read it without problem. This is because we do not read every letter by itself but the word as a whole.

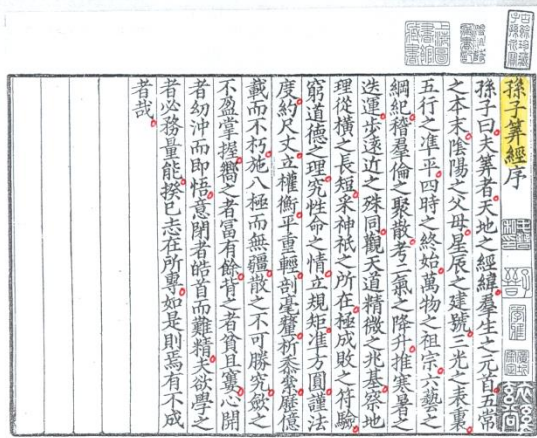
To what extent do you believe in the claim made in this internet meme that went the rounds in September of 2003?

According to a research at an English university, it doesn't matter in what order the letters in a word are, the only important thing is that first and last letter are at the right place. The rest can be a total mess and you can still read it without problem. This is because we do not read every letter by itself but the word as a whole.

The reason for showing you this experiment is not to discuss with you the topic of cognition in reading.

I just want to borrow this example as an analogy to indicate that in order to **understand** and **appreciate** mathematics it is not imperative that you should follow every single step of the technical content in detail meticulously.

You need only get an impression of how things **hang together as a whole**.



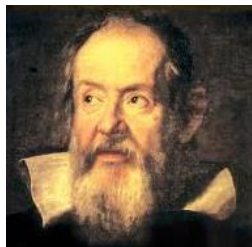
孫子曰：夫算者，天地之經緯，
群生之元首；五常之本末，陰陽
之父母；星辰之建號，三光之表
裏；五行之準平，四時之終始；
萬物之祖宗，六藝之綱紀。

稽群倫之聚散，考二氣之降升；推寒暑之迭運，
步遠近之殊同；觀天道精微之兆基，察地理從橫
之長短；采神祇之所在，極成敗之符驗；窮道德
之理，究性命之情。立規矩，準方圓，謹法度，
約尺丈，立權衡，平重輕，剖毫釐，析黍絛；

歷億載而不朽，施八極而無疆。散之不可勝究，
斂之不盈掌握。嚮之者富有餘，背之者貧且窶；
心開者幼沖而即悟，意閉者皓首而難精。夫欲學
之者必務量能揆己，志在所專。如是則焉有不成
者哉。

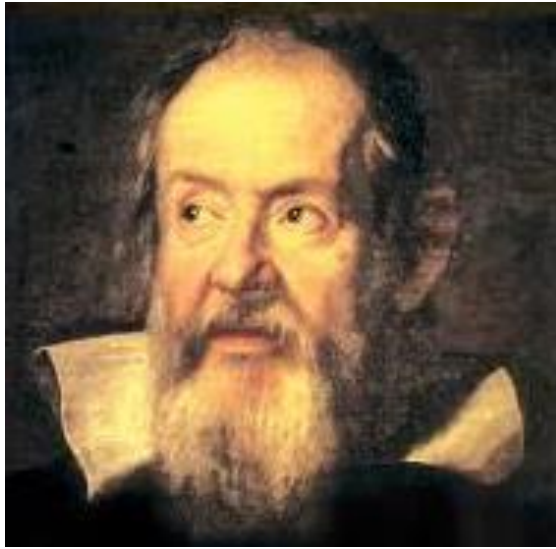
《孫子算經》(公元四/五世紀)

“Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and reads the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth.”



Il Saggiatore (The Assayer)

**Letter to the Illustrious and Very Reverend Don
Virginio Cesarini from Galileo Galilei (1623)**



Galileo Galilei
(1564-1642)

HOW (MUCH) rather
than **WHY**?

Power of
Mathematics

[a quantitative rather than a
qualitative description]

Curiosity

Imagination

**Disciplined and Critical
Thinking**

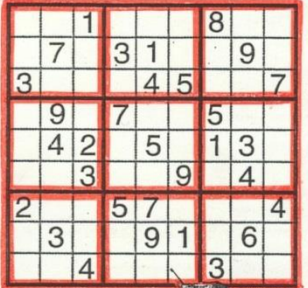
(precision in
mathematics as well as
in **words**)

英報形容如「21世紀扭計骰」

前港港寫撤歐美數字拼圖熱

一擁有200年歷史的數字遊戲近來風靡歐美民衆，熱鬧之處遠逾當年的「扭計骰」。這種日本人稱爲「數獨」(Sudoku)的數字拼圖遊戲，要求玩家動腦筋，有人認爲它有助於減少患老人癡呆症機會，英國數學家更建議用它作高級學生腦力活動，英國多份報章更在論壇位置刊登遊戲供讀者玩耍，藉以刺激銷量。

數字拼圖「遊戲玩法」



這種數字拼圖遊戲於18世紀初期由一名德國數學家所發明。其後一些多出版內閣深受歡迎。美國一本雜誌在20年前首次推出它的魅力，日本人進一步增加難度，並把它命名爲「數獨」，並將其推廣到雜誌出版。曾任香港高等法院法官的前英國人羅傑斯 (Wayne Gould) 在與內英編《泰晤士報》推廣這項遊戲，讓其銷路大增。這是在英國報紙熱潮。

英報刊「數獨」搶銷量

據悉，多份全歐報章紛紛刊登它，其原文在《衛報》周刊在雜誌宣佈了「數獨」(Sudoku) 的報章。《泰晤士報》則大字標題稱道：「「手腦遊戲」：每個人都該試試的遊戲，現在出現在你的手機上了。」《獨立報》亦在版刊載一則標題的「數獨」遊戲，並在內頁再刊載不同難度的版本。

命名「數獨」意指由個位數組成

「數獨」玩法簡單，遊戲主體是一個由九個九格排列成的大正方形，每一行每一列都由九個小方格組成。遊戲通常由第一至九個數字填上1到9的不同數字，要求每橫每直每格，橫向與大正方形每一行每一列及每九格橫向必須包括1到9的每一個數字。

數獨雜誌引入課堂

英國明報最近有人討論「數獨」：首本關於「數獨」的月刊已於周前推出；多本關於「數獨」的書在市場上出售；電視台亦正在播映相關的電視節目；英

府教育司的《教師》雜誌更建議把「數獨」引入課堂，有助學生腦力活動。就英國廣播公司新聞頻道也對節目談到許多與「數獨」遊戲。《觀察家報》形容，「數獨」就如「21世紀的扭計骰」。扭計骰在1961年推出後，曾風靡全球，至今尚流行。

智力遊戲迷 一見傾情努力研究

「數獨」遊戲在20年前便在日本大受歡迎。在真正合上「全球性」的，卻是於香港高等法院法官羅傑斯(韋恩)高橋德去年把這遊戲帶到英國《泰晤士報》，成功掀起熱潮。

高橋德曾完全一份英國地方報紙上刊登他設計的一個「數獨」遊戲，獲得熱烈迴響。去年底，《數獨》雜誌在英國出版，他擔任編輯。他說：「我(當時)正處於倫敦前往香港，像他(高橋德)一樣，在沒有通知下走進《泰晤士報》，讓這遊戲出不同「數獨」版本的電腦程式。他去年接受《泰晤士報》訪問時說：「我輸入了2.5萬個獨立數字拼圖，讓全球電腦化(電腦程式)解答，對我來說，這是個全新起點，在正常情況下，你不會想辦法將數字填到電腦程式，對不對？」



花8年時間 設計遊戲電腦程式

高橋德本身是一名智力遊戲迷。1997年，他在編退休前到東京度假，在銀座一書店滿腦子念起一本智力遊戲書。羅傑斯「數獨」遊戲吸引，「我一看到那本小書就買了，但後來發現這門「高橋德隨後花了6年時間，設計出一個能產生出不同「數獨」版本的電腦程式。他去年接受《泰晤士報》訪問時說：「我輸入了2.5萬個獨立數字拼圖，讓全球電腦化(電腦程式)解答，對我來說，這是個全新起點，在正常情況下，你不會想辦法將數字填到電腦程式，對不對？」

日人改良 遊戲重見天日

「數獨」遊戲其實已有逾200年歷史。1783年，瑞士數學家歐伊勒(Leonhard Euler)發明了它的前身「拉丁方格」(Latin Squares)。「拉丁方格」與「數獨」的唯二不同之處，是它沒有部分分成九宮格。1970年代末期，英國益智雜誌《The Puzzle Magazine》開始刊載「拉丁方格」

，稱爲「數字拼圖」。日本益智雜誌的編者金成廣相信這項遊戲有助於擴大雜誌的銷售群，決定把它引入日本。他說：「數獨」的奇妙之處在於它非常令人著迷，雖然它很簡單，但如果你會玩小說讀到發悶，也不會覺得「數獨」無聊。這日本人改裝後稱爲「數獨」，很快在日本掀起熱潮。 星期日電訊

Financial Times

2005.05. 28/29

Count me in on the Sudoku number puzzle craze

I first came across Sudoku a couple of Saturday mornings ago, when I wandered down to the corner shop to buy a weekend newspaper, writes Stephen Pincock.

The folks who write excellent "local mixed business" stock most of Britain's plethora of daily national papers, which they arrange in convenient piles on a low wooden platform near the front door. Bending down to grab the FT (naturally) and some other rag, I noticed the word "Sudoku" on the other paper's front page.

It was, unsurprisingly, in a prominent spot at the top right-hand corner of the page. Glancing along the pile, I saw the word crop up above the fold on a surprising number of the other papers. One touted its handcrafted "classic Sudoku", another had three puzzles of varying complexity, while yet another claimed to have been the first to introduce Sudoku to the UK. What, I thought, are these things?

It didn't take long to find that they are number puzzles laid out on a nine by nine grid of squares, some of which contain numbers between one and nine. The idea is to fill in the blank squares in such a way that each row, each column and every three-by-three square within the larger grid contains every number between one and nine, once only.

A quick search on the internet revealed that the Sudoku fad came to Britain via Japan, where the puzzles have been popular since the 1980s. The "su" in the name stands for number, while "oku" connotes singular. In Japan, the name Sudoku is the copyright of Nikoli publishing, although the puzzle seems to have been published first in the US, where it is called the Number Place puzzle.

In fact, as it turns out, you can trace the origins of Sudoku back to something invented in the 18th century by one of history's greatest mathematicians, Leonard Euler. Euler was a Swiss-born child prodigy who went on to become one of the most prolific mathematicians of all time. In 1782, he introduced the idea of "Latin squares", which he called a "nouveau espèce de carrés magiques" - a new kind of magic squares. Latin squares are grids of any size in which each number appears once in each column and each row, just as in Sudoku puzzles. Regardless of their heritage, I've learned that Sudoku can be complicated. It didn't take too many hours after that trip to the newsagents before my wife was wondering whether she was becoming a "Sudoku widow".

The rules are simple, although they can become fiendishly difficult as the number of given numbers drops. But the odd thing is that newspapers make a point of saying that the puzzles require no mathematics.

The assumption seems to be that maths is nothing more than arithmetic - addition, subtraction, multiplication and division. This reflects a complete misunderstanding of what the subject is all about, says Charles Leedham-Green, professor of pure mathematics at Queen Mary, University of London. "One thing that mildly irritates me is this idea that because you don't have to add the numbers up in Sudoku then it's not mathematics."

In fact, the process involved in figuring out how to solve number placement puzzles is very mathematical, according to Leedham-Green. "Thinking of a way to tackle a Sudoku is a mathematical problem. Obviously, no one thinks it is high mathematics, but the process of coming up with the tricks to solve these puzzles is mathematical."

Marcus du Sautoy, professor of mathematics at the University of Oxford, agrees. In fact, he sees parallels between the thrill of beating a Sudoku puzzle and the excitement of mathematical breakthroughs. "I think the buzz people get from finishing them is the same kind of buzz I get from mathematics," he says.

The thing is, mathematicians don't spend their days working through page after page of mind-numbing arithmetic. Rather, they bend their minds to finding ways to solve problems that are often deeply conceptual. They approach these challenges using the kind of strategic thinking you need to employ in tackling Sudoku. You might not know it, but the deci-

... make in finishing them fall into the category of heuristic, a discipline of solving problems using intelligent choices when applying a formula is impossible. There are other maths terms that might apply to Sudoku as an exercise in "graph colouring", and consider a strategy for tackling it as a "backtrack search".

The puzzles themselves have even triggered their own mathematical conundrum - the number of possible initial arrangements is directly unknown, despite the fact that mathematicians have figured out the number of possible variants of a Latin square that's nine by nine.

I'm not trying to hassle you with terminology here, but to point out that Sudoku might teach us something unexpected about maths and mathematics. Put aside the prejudices developed during double-maths on a sunny Friday afternoon and consider this idea - maths might actually be fun.

stephen.pincock@journalist.co.uk

The Su-do-ku Craze

Count me in on the Sudoku number craze

Stephen Pincock

Financial Times

May 28, 2005.

Sudoku in Puzzles' Corners of newspapers

Bridge

East dealer. Both sides vulnerable.

NORTH
 ♠ Q 10 3
 ♥ 7 4
 ♦ J 8
 ♣ A Q 10 8 4 3

WEST
 ♠ A 7 2
 ♥ A Q 8 6 3
 ♦ 10 7 4
 ♣ 9 2

EAST
 ♠ J 8 6 5 4
 ♥ J 9
 ♦ 9 6 5 2
 ♣ K 5

SOUTH
 ♠ K 9
 ♥ K 10 5 2
 ♦ A K Q 3
 ♣ J 7 6

The bidding:
 East South West North
 Pass 1NT Pass 3NT

Opening lead: six of hearts
 This deal from a team contract demonstrates one of the more subtle areas of sound defensive play. The final contract at both tables was three notrump, but only one of the defenders found the way to derail declarer.

At the first table, West led the heart six. Declarer won East's jack with the king and led the jack of clubs, on which West played the nine, dummy the three and East the king. East then returned the heart nine, ducked by South, whose only real chance was to hope the opposing hearts were blocked.

West followed with the three to the second heart lead, leaving East in a quandary as to which suit to play next. With virtually nothing to go on, East shifted to a diamond, whereupon declarer ran off four diamonds and five clubs to finish with 10 tricks. Of course, had East returned a spade at trick three, West would have collected a spade and three more hearts for a two-trick set.

Though it might appear that East simply misguessed, there was a better way available, as the defence at the second table demonstrated. Here, too, West led the heart six to the jack and king, and declarer took a losing club finesse. But when East then returned the heart nine and declarer ducked, West played the eight.

This gave East food for thought. While it was possible that West had started with just the A-Q-8-6 of hearts (which would in turn mean that declarer began with five hearts), it was also possible that West started with five hearts and had played the eight as a suit-preference signal asking for the return of the higher-ranking side suit, spades.

Since this was far better than a total shot in the dark, East shifted to a spade, and South very quickly went down two.

Sudoku

	8	6	5	3	9		
		8					
7	3	2		6			
		4	1			6	
1	7		3		8	4	
2		5	9				
	6	1		5	3		
			7				
8	2	9	5		4		

Complete the grid so that each row, column and 3x3 box contains every digit from 1 to 9, inclusive.
 Difficulty: ★★★★★

Yesterday's solution:

5	6	1	7	4	2	3	8	5
2	3	8	1	6	9	4	7	8
3	4	9	5	3	1	2	6	7
9	6	1	7	4	2	5	8	3
1	5	3	6	8	4	7	9	2
7	8	5	2	9	6	3	1	4
6	5	2	4	8	1	7	9	3
4	9	6	3	7	5	1	2	8
3	7	2	9	5	8	6	4	1

Kokonotsu Supersudoku

	7			3				
3	6	2		1				
	6	2		9				
1	4	9		5	8	7		
	9	4		1				
7	5			3		4		
	1			7				

Complete the grid so that each row, column and 3x3 box and the two centre diagonal lines contains every digit from 1 to 9, inclusive.
 Difficulty: ★★★★★

Yesterday's solution:

5	8	1	6	7	2	9	3	4
6	4	3	1	5	9	8	7	2
9	2	7	8	3	4	6	5	1
4	6	2	3	8	1	7	9	5
8	3	9	5	2	7	4	1	6
7	1	5	9	4	6	3	2	8
2	9	8	7	6	5	1	4	3
1	5	6	4	3	2	9	7	8
3	7	4	2	1	8	5	6	9

Target

T	O	C
E	P	C
A	M	D

How many words of four letters or more can you make from the letters shown here? In making a word, each letter may be used once only. Each must contain the centre letter and there must be at least one nine-letter word. No plurals.

Today's target: 20 words good, 30 very good; 39 words excellent. See solution tomorrow.

BRIDGE PAUL MENDELSON

To set up a big suit in dummy, you usually require multiple entries and, in a suit contract, your trumps may provide the key access.

South opened a strong 2S and was raised to 4S, which denied any ace but showed values. Unable to investigate scientifically, South punted the slam. West led 10W and South could see immediately that he has a diamond to lose as well as a club finesse to take. At the

CHESS LEONARD BARDEN

The annual London Classic at Olympia introduced a new format this year which sparked a rash of entries.

Before the main six-player elite event there was a regularity (one hour games) where the world's best competed with amateurs.

The 400-strong field saw some early rocky passages for the grandmasters. The Turk student James Aughton (E) was the first to be eliminated. Nakamura (I) was the first to be eliminated in the 1000+ category.

Next round, the American won a chess match. H Nakamura (I) was the first to be eliminated. Nakamura (I) was the first to be eliminated.

City investment financier All Mortzans missed the chance for the result of his life here when he played 1 E2H1 Q6e here and lost to the world No 2. How could White have defeated Fabiano Caruana?

Solution, back page

CROSSWORD 14,804 SET BY MAGWITCH

ACROSS

- 1 Reverse record names (7)
- 2 Ken agency backs impressive chessboard (6)
- 3 10 extremely beguilingly defined new movement against local government (10)
- 4 Not work by copper turning into management for forcible removal (6)
- 5 Pay for advertising in time to make an impression (6)
- 6 Since it not vitally put on the table without testimony (6)
- 7 Ship to Europe in cabin (6)
- 8 Not sure if it should sit top level (6)
- 9 Failing to pay attention to nightwear with embellishment (6)
- 10 Select the best among mediocre candidates (6)
- 11 Worth money to return not completely satisfied (6)
- 12 Not tribute to soldier buried in unpopulated forest (6)
- 13 Factor that was decisive to abandon Europe (6)
- 14 Right choice by checking against in resolution (6)
- 15 Equipment developed by the French in service (7)
- 16 City's top teacher in England was the last released (7)

DOWN

- 1 See computer-generated image to go after man (7)
- 2 Swearing before politician with nothing to confess (6)
- 3 Another ahead in scoring means reverse (3,2)
- 4 Appreciation by doctor argued to accept carrying on (6)
- 5 4000-year-old vehicle manufacture over introduction of background (6)
- 6 Entry three centuries famed and different than martial (6)
- 7 See computer-generated image to go after man (7)
- 8 Call for quarters international support for prisoner (7)
- 9 Handwritten manuscript after banking system out the (6)
- 10 Single chess move in a list of all (6)
- 11 Definition of amnesia in chronic (6)
- 12 Right letter appears at end of drama about objectives (7)
- 13 Usually applied to the dolls, but (7)
- 14 Period before resistance to over (7)
- 15 Improving understanding of operations leader by providing night of reply (6)
- 16 Prayer intended to an antislavery assembly (6)

BRIDGE PAUL MENDELSON

table, South drew trumps and led ♣. When East played low, he ran the ♣ and this lost to West's ♠. South's fourth club lost also, leaving him two down. Is there a superior line?

Suit establishment should always be at the forefront of your mind and, here, dummy's diamonds can be set up. South wins trick one and leads 7W. Whoever card he plays from dummy he loses. He regains the lead by playing 10W to ♠ and leads low diamond from hand, which he trumps high. ♠ now led to dummy's ♠, as a second diamond ruff. This pulls out East-West's remaining cards, leaving dummy's three diamonds winners. South reaches the table by playing his carefully preserved ♠ to dummy's ♠. Now, all three club loss can be thrown on the diamond winners.

These suit establishments occur regularly. If you can master them, your scores will improve dramatically.

Crossword 14,804

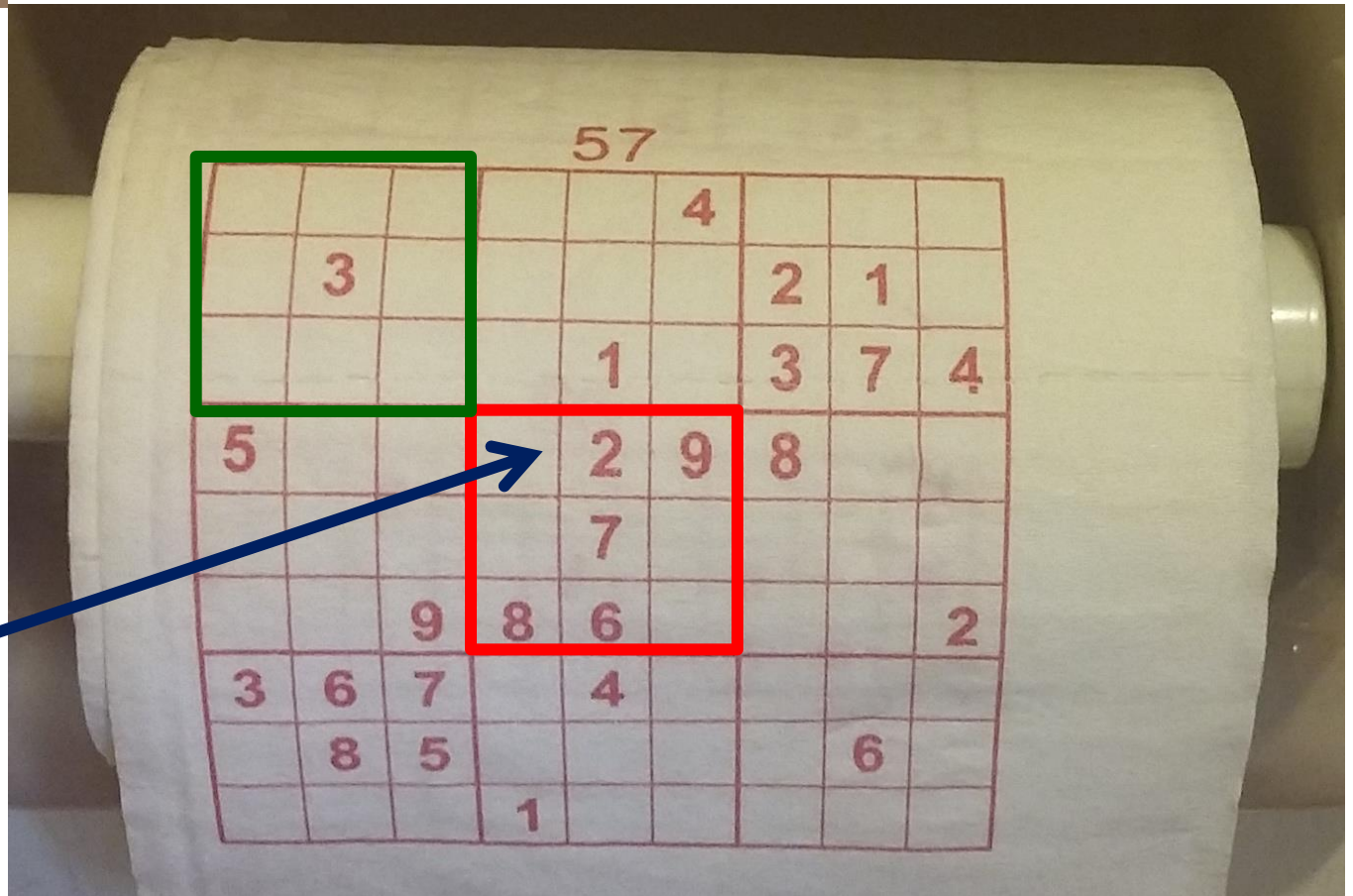
Solution 14,803

Solution 14,972

South China Morning Post, May 21, 2015

Still popular after the puzzle was introduced to the public readership a decade ago in 2004!

Sudoku in Toilet



This cell
can only be
filled with
3 or **4**, because
1 and 5
are not
allowed.

Count me in. The real puzzle behind Sudoku is the idea that maths doesn't come into play. Well, the number crunchers will have the last laugh.

Stephen Pincock

Financial Times, May 28, 2005.

"One thing that mildly irritates me is this idea that because you don't have to add the numbers up in Sudoku then it's **not** mathematics."

Charles Leedham-Green,

Professor of Pure Mathematics

at Queen Mary College, University of London.

P = PHILOSOPHY MASTER
J = MONSIEUR JOURDAIN

Moliere's *Le Bourgeois gentilhomme*
(October 14, 1670)

P : What isn't verse is prose, and what's not prose is verse.

J : And this, the way I speak. What name would be applied to the --

P : The way you speak?

J : Yes.

P : Prose.

J : It's prose?

P : Decidedly.

J : Oh really? So when I say: "Nicole bring me my slippers and fetch my nightcap," Is that prose?

P : Most clearly.

J: **Well, what do you know about that ! These forty years now, I've been speaking in prose without knowing it !** How grateful am I to you for teaching me that ! ...



Molière
[Jean-Baptiste Poquelin]
(1622-1673)



Sudoku, as a puzzle of pure logical deduction, provides a good means to study **the underlying mental processes in naïve individuals, that is, those who have had no training in logic.**

How much ability in pure deductive reasoning do naïve individuals really possess?

N.Y. Louis Lee, G.P. Goodwin, P.N. Johnson-Laird, The psychological puzzle of Soduku, *Thinking & Reasoning*, 14(4) (2008), 342-364.

- ❖ Logically naïve individuals have the competence to make deductions about **abstract matters**, and they **enjoy** exercising this ability.
- ❖ The **teaching of mathematics** can be more effective when it focuses on **abstract matters** rather than on **concrete everyday examples**

N.Y. Louis Lee, G.P. Goodwin, P.N. Johnson-Laird, The psychological puzzle of Sudoku, *Thinking & Reasoning*, 14(4) (2008), 342-364.

As Piaget recognised, our ability to make deductions about **abstract matters remote from our mundane life is a **fundamental human characteristic**, and one that is **essential to intellectual progress.****

N.Y. Louis Lee, G.P. Goodwin, P.N. Johnson-Laird, The psychological puzzle of Sudoku, *Thinking & Reasoning*, 14(4) (2008), 342-364.

Good points about mathematical games and puzzles

- ❖ **Nurturing of**
 - observation**
 - concentration**
 - patience**
 - curiosity**
 - flexible thinking**
- ❖ **Training of**
 - logical thinking**
 - space visualization**
 - systematic analysis**
 - meticulous working**

Genuine
“learning
in pleasure
[愉快學習]”

嬰兒生無石師而能言

與能言者處也。 莊子·外物（公元前四世紀）

**[When a child is born, it needs no
great teacher ;
nevertheless it learns to talk as it
lives with those who talk.]**

Zhuangzi (Chuang Tzu)

Book 26 : Affected from Outside

(4th century B.C.E.)

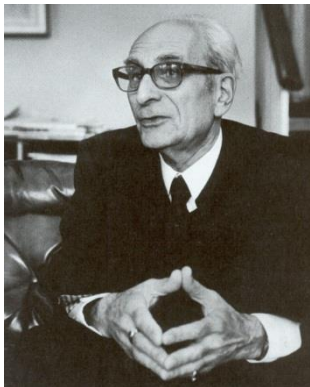
We don't stop playing because we grow old; we grow old because we stop playing.

(Variably attributed to Benjamin Franklin, Oliver Wendell Holmes (Sr. or Jr.), Herbert Spencer, George Bernard Shaw)

“Groos [Karl Groos] well says that children are young because they play, and not vice versa; and he might have added, men grow old because they stop playing, and **not** conversely, for **play** is, at bottom, **growth**, and at the top of the intellectual scale it is the **eternal type of research from sheer love of truth.**”

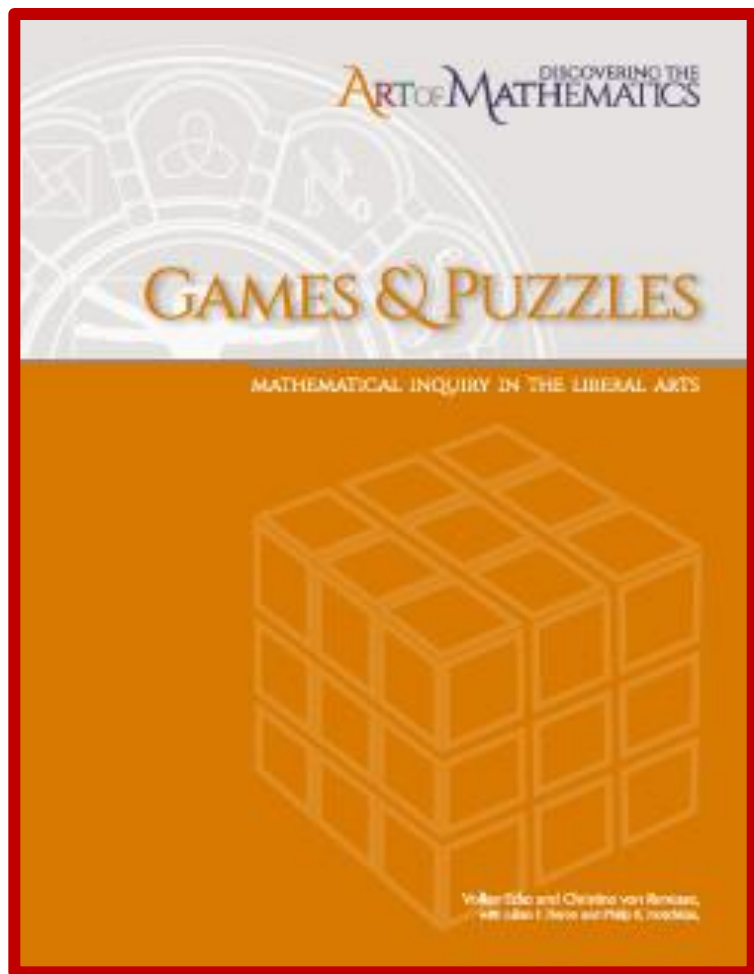
*G. Stanley Hall, **Adolescence: Its Psychology and its Relations to Physiology, Anthropology, Sociology, Sex, Crime, Religion and Education** (1904)*

“ ... the student who chooses the teaching profession does not bid farewell to the world of childhood: on the contrary he is trying to remain within it.”



Claude Lévi-Strauss
(1908 - 2009)

要保持童心！



Games and Puzzles

Volker Ecke and Christine von Renesse, with Julian F. Fleron and Philip K. Hotchkiss,

2015 ; current version 2018 .

Section 3.2 : Radon/Kaczmarz Puzzles

15 15 15 15

15

15

15

15

You are given the three row sums, three column sums and the two diagonal sums, fill in the nine cells in the 3 x 3 grid with suitable positive integers chosen from 1 to 9.

2	7	6	15
9	5	1	15
4	3	8	15
15	15	15	15

You are given the three row sums, three column sums and the two diagonal sums, fill in the nine cells in the 3 x 3 grid with suitable positive integers chosen from 1 to 9.

5	5	5	15
5	5	5	15
5	5	5	15
15	15	15	15

You are given the three row sums, three column sums and the two diagonal sums, fill in the nine cells in the 3 x 3 grid with suitable positive integers chosen from 1 to 9.

4	6	5	15
6	5	4	15
5	4	6	15
15	15	15	15

You are given the three row sums, three column sums and the two diagonal sums, fill in the nine cells in the 3 x 3 grid with suitable positive integers chosen from 1 to 9.

3	5	7	15
9	5	1	15
3	5	7	15
15	15	15	15

You are given the three row sums, three column sums and the two diagonal sums, fill in the nine cells in the 3 x 3 grid with suitable positive integers chosen from 1 to 9.

7	2	6	15
4	5	6	15
4	8	3	15
15	15	15	15

You are given the three row sums, three column sums and the two diagonal sums, fill in the nine cells in the 3 x 3 grid with suitable positive integers chosen from 1 to 9.

2	7	6
9	5	1
4	3	8

5	5	5
5	5	5
5	5	5

4	6	5
6	5	4
5	4	6

3	7	5
7	5	3
5	3	7

2	8	5
8	5	2
5	2	8

1	9	5
9	5	1
5	1	9

3	5	7
9	5	1
3	5	7

4	5	6
7	5	3
4	5	6

3	6	6
8	5	2
4	4	7

There are altogether **41 solutions** to the puzzle, falling into essentially **9 types** with the remaining ones obtained via rotation or reflection.

3	9	3
5	5	5
7	1	7

6	6	3
2	5	8
7	4	4

4	8	3
4	5	6
7	2	6

7	5	3
1	5	9
7	5	3

5	7	3
3	5	7
7	3	5

There are **9 unknowns** and **12 equations**. Why can't the given conditions pin down the solution?

Even if in addition we are given the **five** *NW-SE* diagonal sums the answer is **still not unique**.

3	9	3
5	5	5
7	1	7

6	6	3
2	5	8
7	4	4

4	8	3
4	5	6
7	2	6

7	5	3
1	5	9
7	5	3

5	7	3
3	5	7
7	3	5

If in addition we are given all the **ten** diagonal sums, then the answer will be **unique**.

15 15 15

15

15

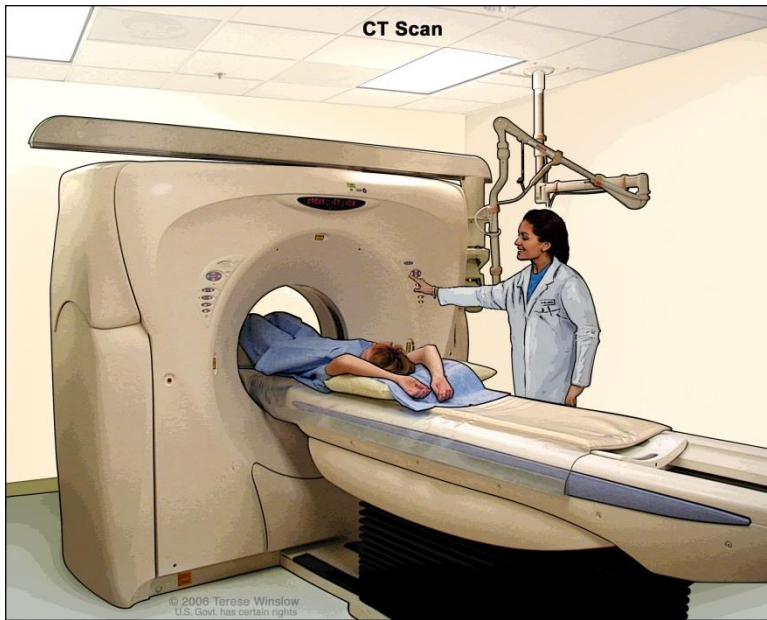
15

15

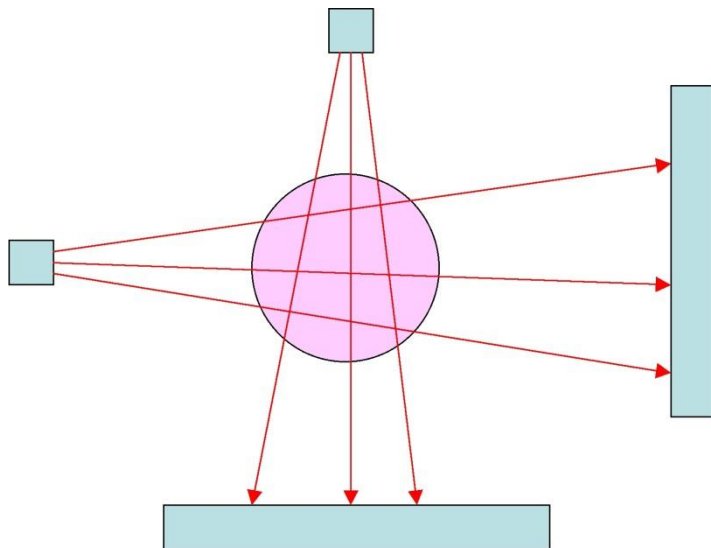
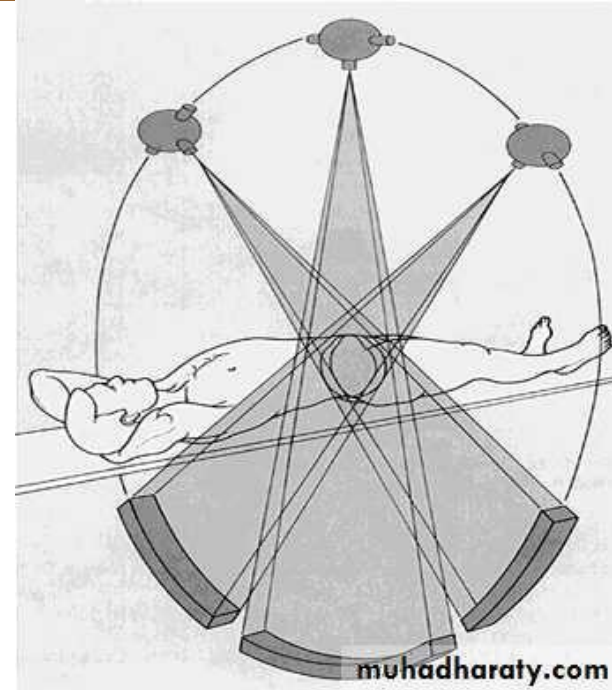
15

In general we are looking at a system of 8 linear equations with 9 unknowns, the rank of the coefficient matrix being equal to 7. Of the 9 unknowns there are 7 pivotal unknowns and 2 free unknowns.

Question: What sort of conditions will guarantee a **unique solution** if one exists?



CT Scan (Computerized Tomography)



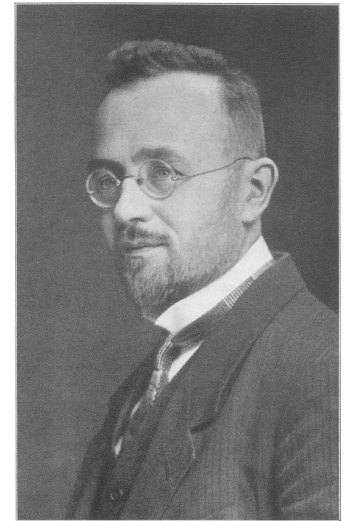
Basically we try to figure out the entries of a large grid knowing the row sums, column sums, diagonal sums, etc.



Kaczmarz's algorithm for solving a system of linear equations, 1937.

Mathematics,

Radon (Inverse) Transform, 1917.



Stefan Kaczmarz

Stefan Kaczmarz
(1895-1940)

J. Radon

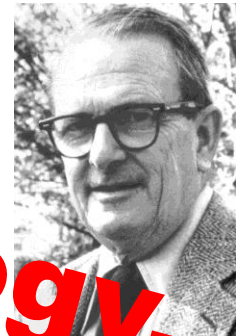
Johann Radon
(1887-1950)



Science, Engineering, Technology,



Nobel Prize for Physiology or Medicine for development of diagnostic technique of X-ray CT (computed tomography), 1979.



Godfrey Newbold Hounsfield
(1919-2004)

Allan McLeod Cormack
(1924-1998)

$$\cos(\pi/4) = 0.7071067\dots$$



707 (1957)



717



727



737



747



757



767

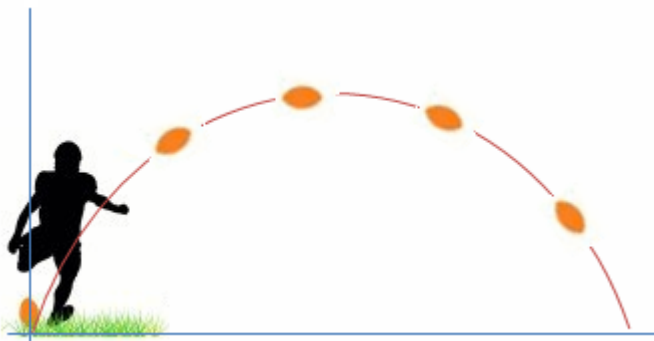


787

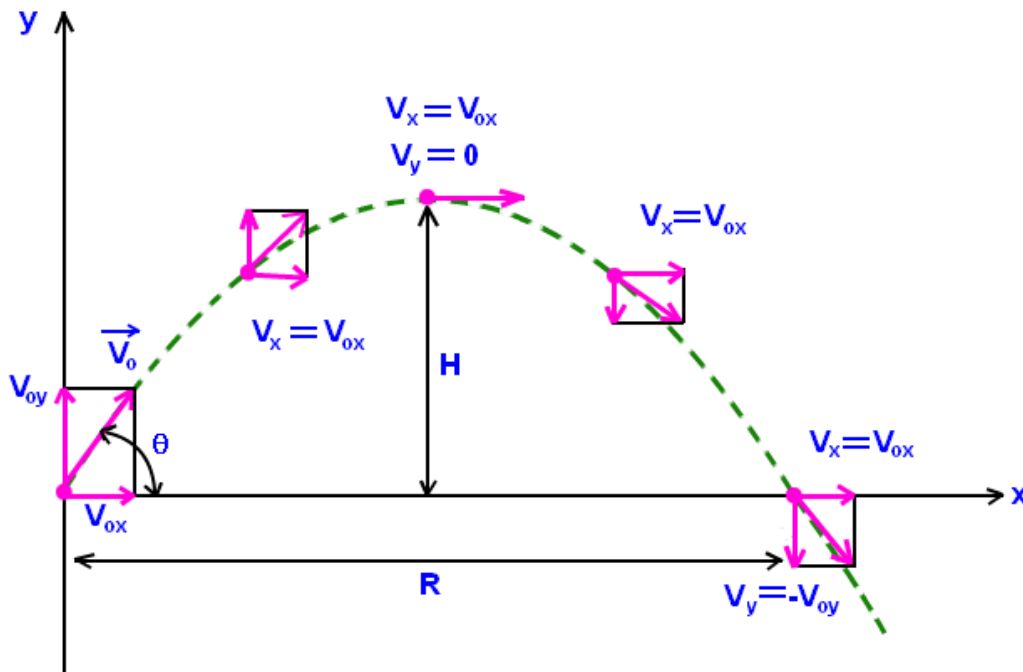


777

**Boeing
Airplanes**



Projectile 拋體運動



<http://ggbtu.be/m1082291>



SMET

Science

Mathematics

Engineering

Technology

It was said that in 2001 Judith Ramaley, then Assistant Director for Education and Human Resources at NSF, thought that **SMET** does not sound as good as **STEM**, so she changed the acronym to **STEM**!

STEM

Science

Technology

Engineering

Mathematics

STEM

Science

Technology

Engineering

Mathematics

STEAM

Science

Technology

Engineering

Arts

Mathematics

STEM

Science

Technology

Engineering

Mathematics

STREAM

Science

Technology

Reading

Engineering

Arts

Mathematics

STEM

Science

Technology

Engineering

Mathematics

*i*STREAM

information science

Science

Technology

Reading

Engineering

Arts

Mathematics

STEM

Science

Technology

Engineering

Mathematics

STREAiM

Science

Technology

Reading

Engineering

Artificial

intelligence

Mathematics

With a **humanistic and caring mind** breed a feeling of humbleness and tolerance by learning from the long history of the human race. **Science is not almighty to command everything.** Instead, we should learn how to live in harmony with **Mother Nature and with others.**

THAMES

TTechnology

Humanities

Arts/**A**rtificial intelligence

Mathematics

Engineering

Science

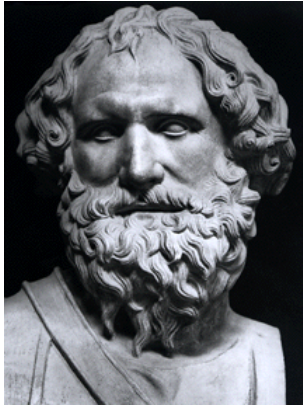
「依我看，**STEM**不是一個學科，也更加不僅是一堆現代科技產品。

STEM是一種**綜合意識**，滲透在不同的學科，以學習和運用**數學**知識及**科學**知識，再以**工程**手段配合現代**科技**，改善生活。

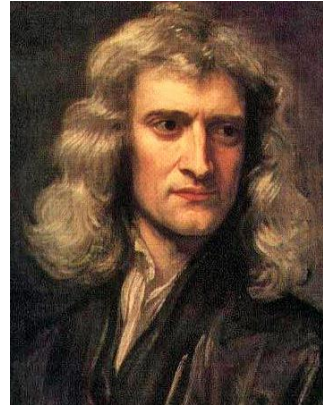
STEM蘊含一種**探索精神**及**思想方式**，揉合了**數學思維**及**科學精神**，通過**實驗、觀察、理論整理**以**尋求知識**，**進而創新**。」

蕭文強, 推薦序,
盧安迪, 《STEM教育與美國》(2018), 頁10.

STEM ANEG



**Archimedes
(287–212 B.C.E.)**



**Isaac Newton
(1642-1727)**

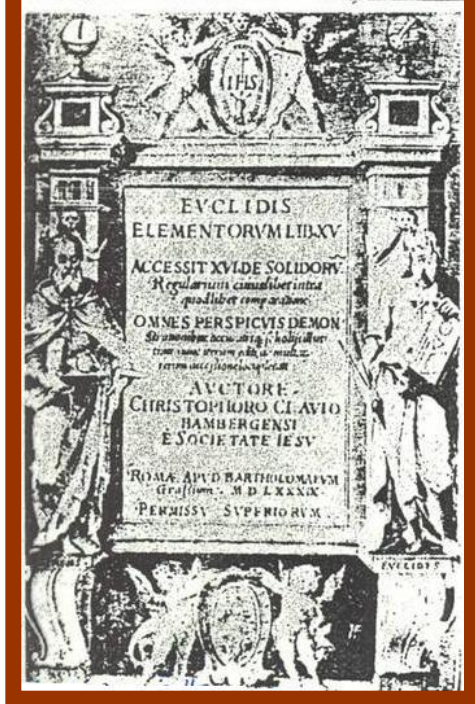


**Leonhard Euler
(1707-1783)**



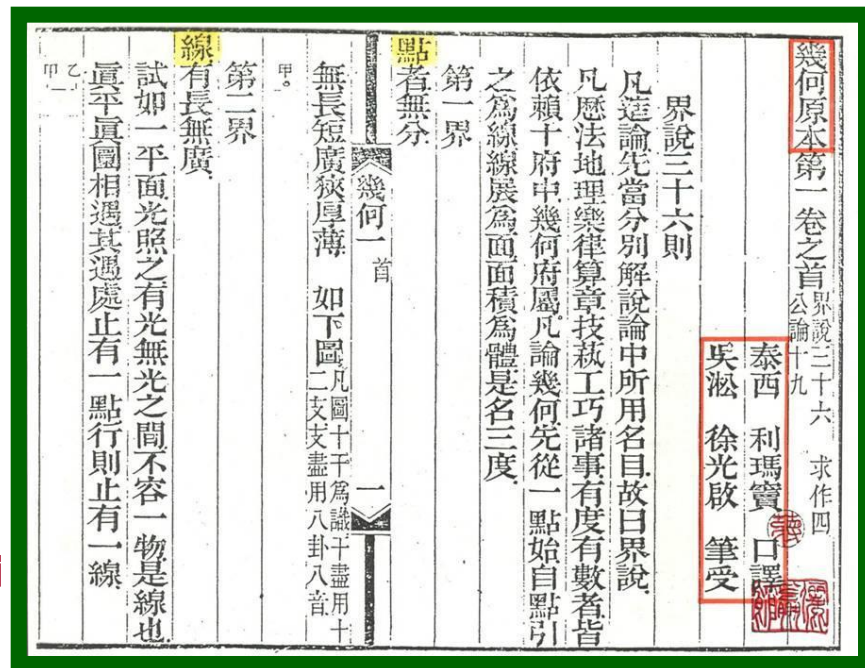
**Carl Friedrich Gauss
(1777-1855)**

**C. CLAVIUS, *EUCLIDIS ELEMENTORUM LIBRI XV*
(1574; 1589)
EUCLID'S *ELEMENTS*
(c. 300 B.C.E.)**



***Jihe Yuanben* [幾何原本]
translated by Matteo Ricci
and XU Guang-qi (1607)**

幾何原本



**Chinese translation
of *Elements*
by Matteo Ricci
[利瑪竇] and
XU Guang-qi
[徐光啟] in 1607**



**Matteo Ricci
(1552-1610)**



**XU Guang-qi
(1562-1633)**

「度數旁通十事」：

「其一（天氣），其二（測量），其三（樂律），
其四（軍事），其五（會計），其六（建築），
其七（機械），其八（輿圖），其九（醫學），
其十（時計）。」

STEM in early
seventeenth century
China

「右十條於民事似為關切。
臣聞之周髀算經云：禹之所以
治天下者，句股之所繇生也。
蓋凡物有形有質，莫不資於
度數故耳。」



徐光啟

XU Guang-qi
(1562-1633)

徐光啟，條議曆法修正歲差疏，1629

What endows mathematics with **its** **power and utility?**

- Logic?
- Computation?
- Use of **symbols**?

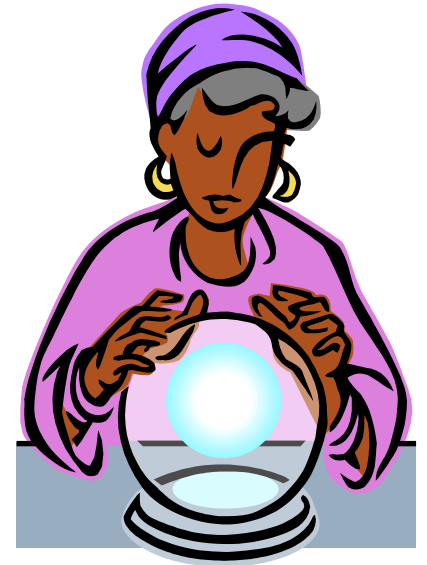
**Why would many people feel uneasy
about the use of symbols?**

- 1) Choose any two-digit number you like.
- 2) Take the sum of the two digits.
- 3) Subtract the sum from the chosen number.
- 4) Find the icon in the list corresponding to the answer you obtain in step (3). Then click the crystal ball.

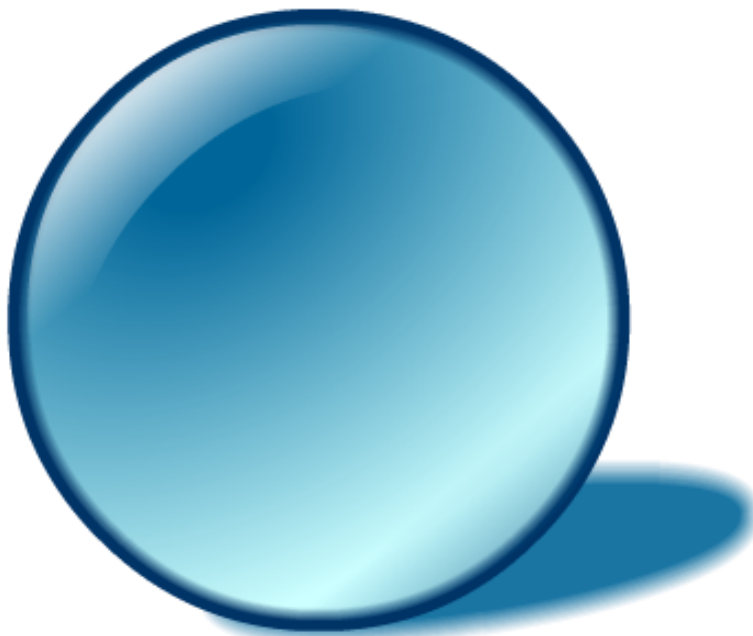
The crystal ball can read your mind by revealing the correct icon!

Example:

$$53 - (5 + 3) = 53 - 8 = 45$$



The Flash Mind Reader



Choose any two digit number, add together both digits and then subtract the total from your original number.*

When you have the final number look it up on the chart and find the relevant symbol. Concentrate on the symbol and when you have it clearly in your mind click on the crystal ball and it will show you the symbol you are thinking of...

99 ☾	79 ♉	59 ♏	39 ☯	19 🔔
98 ❄️	78 ♋	58 ☾	38 ♊	18 ☿
97 ♎	77 ♉	57 ○	37 ♌	17 ⚙️
96 □	76 ♋	56 ♎	36 ☿	16 ♎
95 ☿	75 🌊	55 ♎	35 ♏	15 □
94 †	74 ☾	54 ☿	34 ♎	14 🔔
93 ♋	73 ♉	53 ♋	33 ♋	13 ☿
92 ♎	72 ☿	52 †	32 🔔	12 😊
91 ☾	71 🌊	51 ♉	31 ♎	11 😊
90 ☾	70 †	50 ♉	30 ❄️	10 📖
89 ♎	69 ♎	49 ☾	29 ♏	9 ☿
88 ☯	68 ♎	48 ♋	28 ♎	8 ♉
87 ☾	67 🌊	47 ♎	27 ☿	7 □
86 ♎	66 ♏	46 ♌	26 ♋	6 ♎
85 ⚙️	65 ♎	45 ☿	25 😊	5 🌸
84 ♎	64 □	44 ♋	24 ♎	4 ○
83 ♋	63 ☿	43 ❄️	23 🌊	3 ♎
82 ♏	62 ♌	42 ○	22 🌸	2 ♌
81 ☿	61 📖	41 ⚙️	21 ♋	1 ☾
80 ♉	60 ⚡	40 🌊	20 🌊	0 ☾

* For example if you chose 23: 2+3 = 5. 23 minus 5 will give you your answer.

- 1) Choose any two-digit number you like.
- 2) Take the sum of the two digits.
- 3) Subtract the sum from the chosen number.
- 4) Find the icon in the list corresponding to the answer you obtain in step (3). Then click the crystal ball.

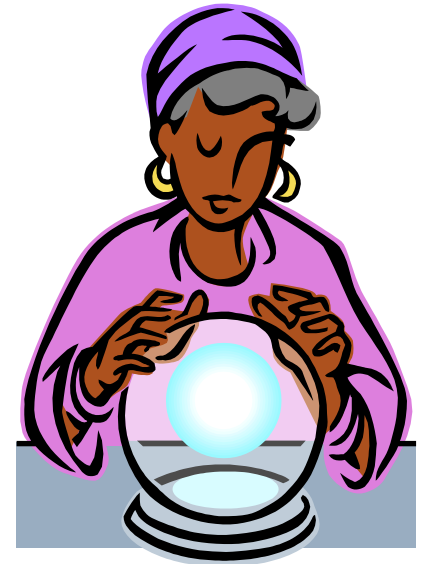
The crystal ball can read your mind by revealing the correct icon!

Example:

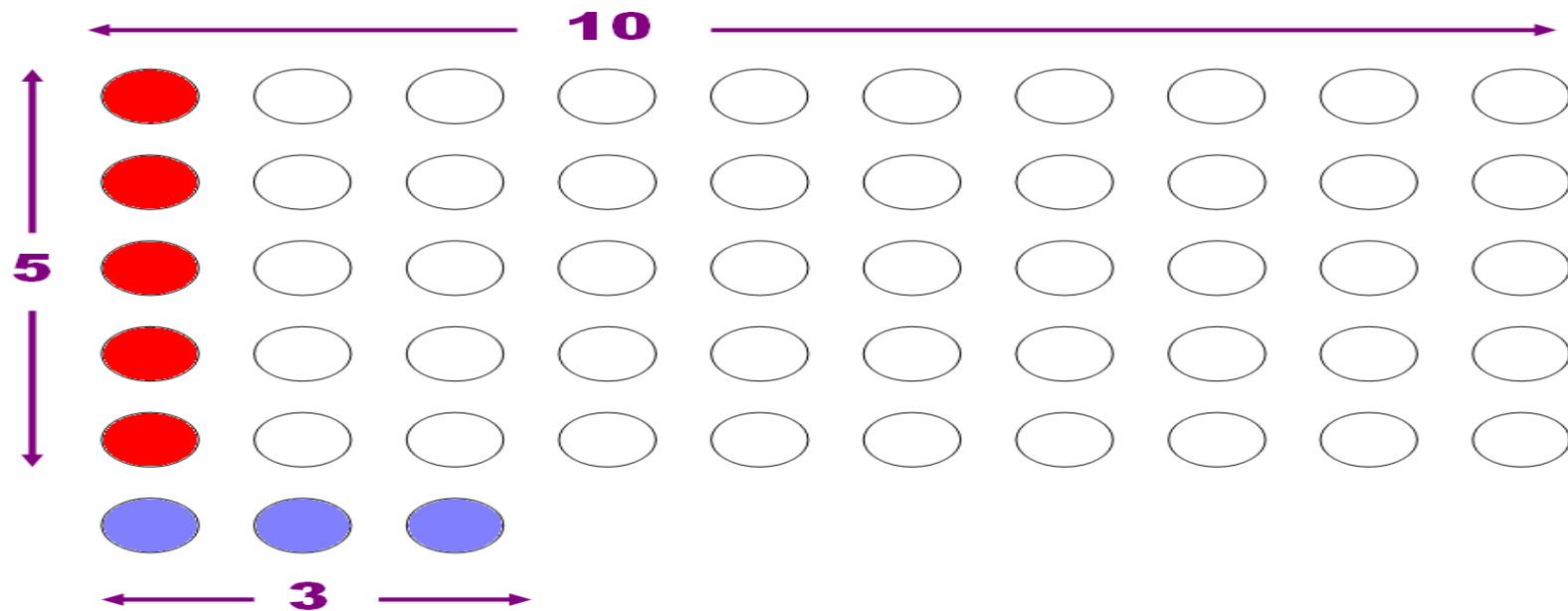
$$53 - (5 + 3) = 53 - 8 = 45$$

$$N = 10A + B$$

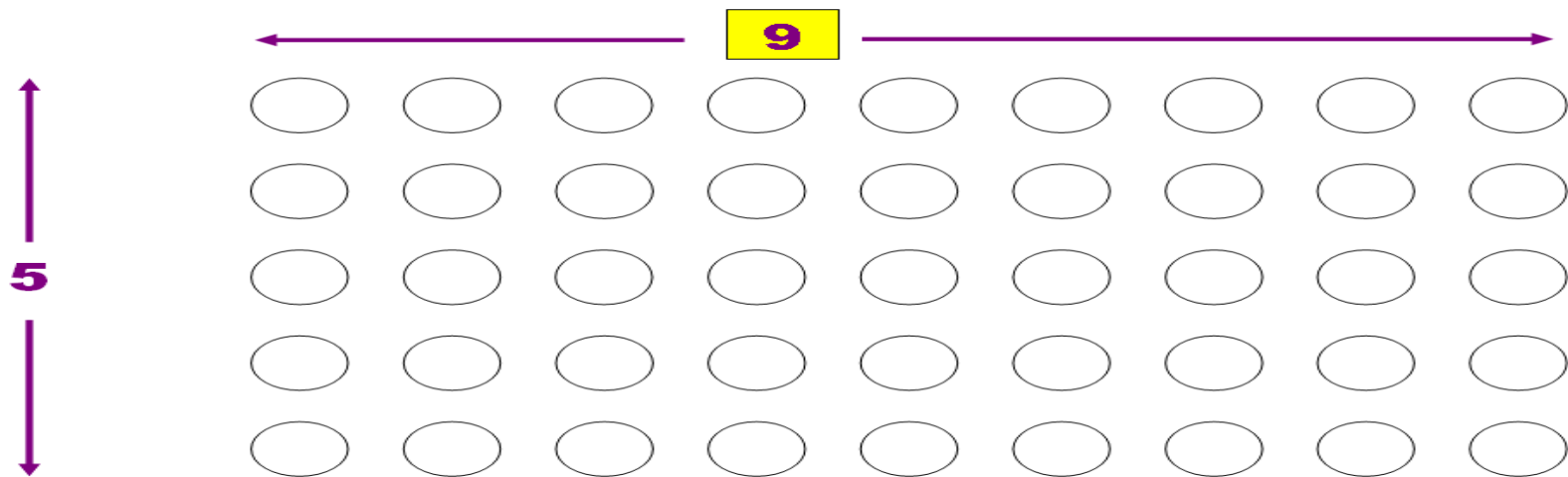
$$N - (A + B) = 10A + B - A - B = 9A$$



$$53 - (5 + 3) = 9 \times 5$$



$$53 - (5 + 3) = 9 \times 5$$





**SYMBOL
ANXIETY**
(符號焦慮症)



**SYMBOL
POWER**
(符號的威力)

**Rhetoric
Algebra**
(言辭代數)



**Syncoptic
Algebra**
(略寫代數)



**Symbolic
Algebra**
(符號代數)

(G.H.F. Nesselmann, 1842)

《九章算術》 [九章算術 Nine Chapters on the Mathematical Art]

ca. 100 B.C.E. — 100 C.E.

Commentary on *Jiuzhang Suanshu*
by LIU HUI [劉徽]

Chapter 9: *Gougu* [勾股]

九章算術卷九

魏劉徽注

算經十書之二

唐開元大夫行太史令上輕車都尉臣李淳風等奉勅注釋

句股以御高深廣遠

今有弦五尺。股四尺。問爲弦幾何。

答曰：五尺。

今有弦五尺。句三尺。問爲股幾何。

答曰：四尺。

今有股四尺。弦五尺。問爲句幾何。

答曰：三尺。

句股短面曰句。長面曰股。相與結角曰弦。故先具此術以見其源也。

術曰：句股各自乘。并而開方除之。卽弦。自乘爲朱方。股自乘爲青方。令出入相補。各從其類。因就其餘。不移動也。合或弦方之。琴開方除之。卽弦也。

又股自乘。以減弦自乘。其餘開方除之。卽句。

臣淳風等謹按此術以句股畢合成弦。琴句方於內。則句短於股。令股自乘。以減弦自乘。餘者卽句。琴也。故開方除之。卽句也。

METHOD

ANSWER

PROBLEM

COMMENTARY (EXPLANATION)

RHETORIC

Diophantus, *Arithmetica* (c. 250 C.E.)

Τετάρθω ὁ μὲν ἐλάχιστος $\overset{\circ}{M} \delta$, ἡ δὲ τοῦ μέσου $\pi^{\lambda} \varepsilon \bar{\alpha} \overset{\circ}{M} \beta$. αὐτὸς ἄρα ἔσται ὁ $\square^{\circ\sigma}$, $\Delta^{\gamma} \bar{\alpha} \varepsilon \delta \overset{\circ}{M} \delta$.

Ἐπεὶ οὖν ἡ ὑπεροχὴ τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχίστου $\gamma^{\sigma\sigma}$ μέρος ἐστίν, καὶ ἔστιν ἡ ὑπεροχὴ τοῦ μέσου καὶ τοῦ ἐλαχίστου $\Delta^{\gamma} \bar{\alpha} \varepsilon \delta$, ὥστε ἡ ὑπεροχὴ τοῦ μεγίστου καὶ τοῦ μέσου ἔσται $\Delta^{\gamma} \gamma^{\chi} \varepsilon \bar{\alpha} \gamma^{\chi}$. καὶ ἔστιν ὁ μέσος $\Delta^{\gamma} \bar{\alpha} \varepsilon \delta \overset{\circ}{M} \delta$. ὁ ἄρα μέγιστος ἔσται $\Delta^{\gamma} \bar{\alpha} \gamma^{\chi} \varepsilon \bar{\alpha} \gamma^{\chi} \overset{\circ}{M} \delta$ ἴσ. $\square^{\circ\sigma}$. πάντα θ^{κ ι} . Δ^{γ} ἄρα $\bar{\iota} \beta \varepsilon \bar{\mu} \eta \overset{\circ}{M} \bar{\lambda} \varepsilon$ ἴσ. $\square^{\circ\sigma}$ καὶ τὸ δ^{ον} αὐτῶν $\Delta^{\gamma} \bar{\gamma} \varepsilon \bar{\iota} \beta \overset{\circ}{M} \bar{\theta}$ ἴσ. $\square^{\circ\sigma}$.

Ἐτι δὲ θέλω τὸν μέσον τετράγωνον ἐλάσσονα εἶναι $\overset{\circ}{M} \bar{\iota} \varepsilon$, καὶ τὴν π^{λ} δηλαδὴ ἐλάσσονος $\overset{\circ}{M} \delta$. ἡ δὲ πλευρὰ τοῦ μέσου ἐστίν $\varepsilon \bar{\alpha} \overset{\circ}{M} \beta$. ἐλάττονές εἰσι $\overset{\circ}{M} \delta$. καὶ κοινῶν ἀφαιρεθειῶν τῶν $\beta \overset{\circ}{M}$, ὁ ε ἔσται ἐλάσσονος $\overset{\circ}{M} \beta$.

Γέγονεν οὖν μοι $\Delta^{\gamma} \bar{\gamma} \varepsilon \bar{\iota} \beta \overset{\circ}{M} \bar{\theta}$ ἴσ. ποιῆσαι $\square^{\circ\sigma}$. πλάσσω $\square^{\circ\sigma}$ τινὰ ἀπὸ $\overset{\circ}{M} \bar{\gamma}$ λειπουσῶν ε τινὰς καὶ γίνεται ὁ ε ἔκ τινος ἀριθμοῦ $\varepsilon^{\kappa\iota\sigma}$ γενομένου καὶ προσλαβόντος τὸν $\bar{\iota} \beta$, τουτέστι τῆς ἰσώσεως τῆς $\varepsilon \bar{\iota} \beta$, καὶ μερισθέντος εἰς τὴν ὑπεροχὴν ἢ ὑπερέχει ὁ ἀπὸ τοῦ ἀριθμοῦ $\square^{\circ\sigma}$ τῶν Δ^{γ} τῶν ἐν τῇ ἰσώσει $\bar{\gamma}$. ἀπῆκται οὖν μοι εἰς τὸ εὐρεῖν τινὰ ἀριθμόν, ὃς $\varepsilon^{\kappa\iota\sigma}$ γενόμενος καὶ προσλαβὼν $\overset{\circ}{M} \bar{\iota} \beta$ καὶ μεριζόμενος εἰς τὴν ὑπεροχὴν ἢ ὑπερέχει ὁ ἀπὸ τοῦ αὐτοῦ $\square^{\circ\sigma}$ τριάδος, ποιεῖ τὴν παραβολὴν ἐλάσσονος $\overset{\circ}{M} \beta$.

$$3z^2 + 12z + 9 = a \text{ square}$$

$$3z^2 + 12z + 9 = (mz - 3)^2$$

$$z = \frac{6m + 12}{m^2 - 3}$$

SYNCOPTIC

1591 François Viète

published *In Artem Analyticem
Isagoge* [Introduction to the
Analytic Art]

- **logistica numerosa**
- **logistica speciosa**

*Quod est, Nullum non problema
solvere.* [There is no problem
that cannot be solved.]

沒有問題是解決不了的！



François Viète
(1540-1603)

angle, jusques a O, en sorte qu'N O soit esgale a NL, la toute OM est z la ligne cherchée. Et elle s'exprime en cete sorte

$$z \propto \frac{1}{2} a + \sqrt{\frac{1}{4} a a + b b.}$$

Que si i'ay $y y \propto - a y + b b$, & qu'y soit la quantité qu'il faut trouver, ie fais le mesme triangle rectangle N L M, & de sa baze M N i'oste N P esgale a N L, & le reste P M est y la racine cherchée. De façon que i'ay $y \propto - \frac{1}{2} a + \sqrt{\frac{1}{4} a a + b b}$. Et tout de mesme si i'auois $x^2 \propto - a x + b$. P M feroit x^2 . & i'auois $x \propto \sqrt{-\frac{1}{2} a + \sqrt{\frac{1}{4} a a + b b}}$: & ainsi des autres.

Enfin si i'ay

$$z^2 \propto a z - b b.$$

ie fais N L esgale à $\frac{1}{2} a$, & L M esgale à b cōme deuãt, puis, au lieu de ioindre les points M N, ie tire M Q R parallele a L N. & du centre N par L ayant descrit vn cercle qui la coupe aux points Q & R, la ligne cherchée z est M Q, oubiẽ M R, car en ce cas elle s'ex-

prime en deux façons, a sçauoir $z \propto \frac{1}{2} a + \sqrt{\frac{1}{4} a a - b b}$, & $z \propto \frac{1}{2} a - \sqrt{\frac{1}{4} a a - b b}$.

Et si le cercle, qui ayant son centre au point N, passe par le point L, ne coupe ny ne touche la ligne droite M Q R, il n'y a aucune racine en l'Equation, de façon qu'on peut assurer que la construction du problemsme proposé est impossible.

La géométrie (1637)

$$z = \frac{1}{2} a + \sqrt{\frac{1}{4} a^2 + b^2}$$

SYMBOLIC



René Descartes
(1596-1650)

1707 Isaac Newton published
Arithmetica Universalis
[Universal Arithmetic]
— work done in 1665-1666



Isaac Newton (1643-1727)

Example 1

$$(X - a)(X - b) = X^2 - (a + b)X + ab$$

$$(X - a)(X - b)(X - c) \\ = X^3 - (a + b + c)X^2 + (ab + bc + ca)X - abc \\ \text{etc}$$

$$(X - a_1)(X - a_2) \cdots (X - a_n) \\ = X^n + c_{n-1}X^{n-1} + \cdots + c_1X + c_0$$

 **easy multiplication**



Does every equation of degree n have n roots?

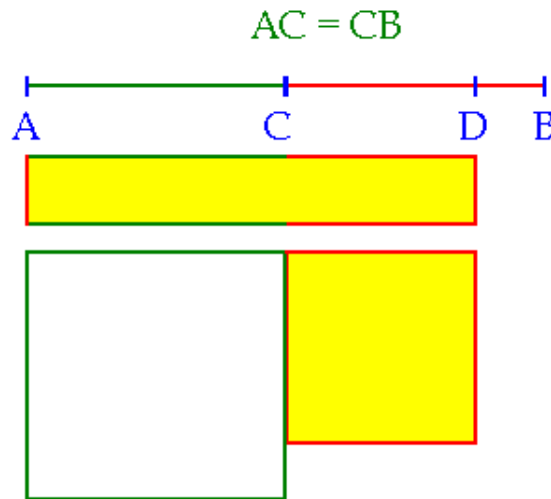
YES!

This result is known as the **Fundamental Theorem of Algebra**, which was referred to in the works of a number of mathematicians, from **A. Girard (1629)** to **L. Euler (1742)**, **J. L. R. D'Alembert (1748)** and **C. F. Gauss (1799)**.

Example 2

Euclid's *Elements*, Book II, Proposition 5 (c. 300 B.C.E.)

The rectangle contained by AD, DB together with the square on CD is equal to the square on AC.



In symbols, $(a - b)(a + b) + b^2 = a^2$,
or $(a - b)(a + b) = a^2 - b^2$.

**Algebra is generous, she often
gives more than is asked of her**
[L'algèbre est généreuse, elle
donne souvent plus qu'on lui
demande].

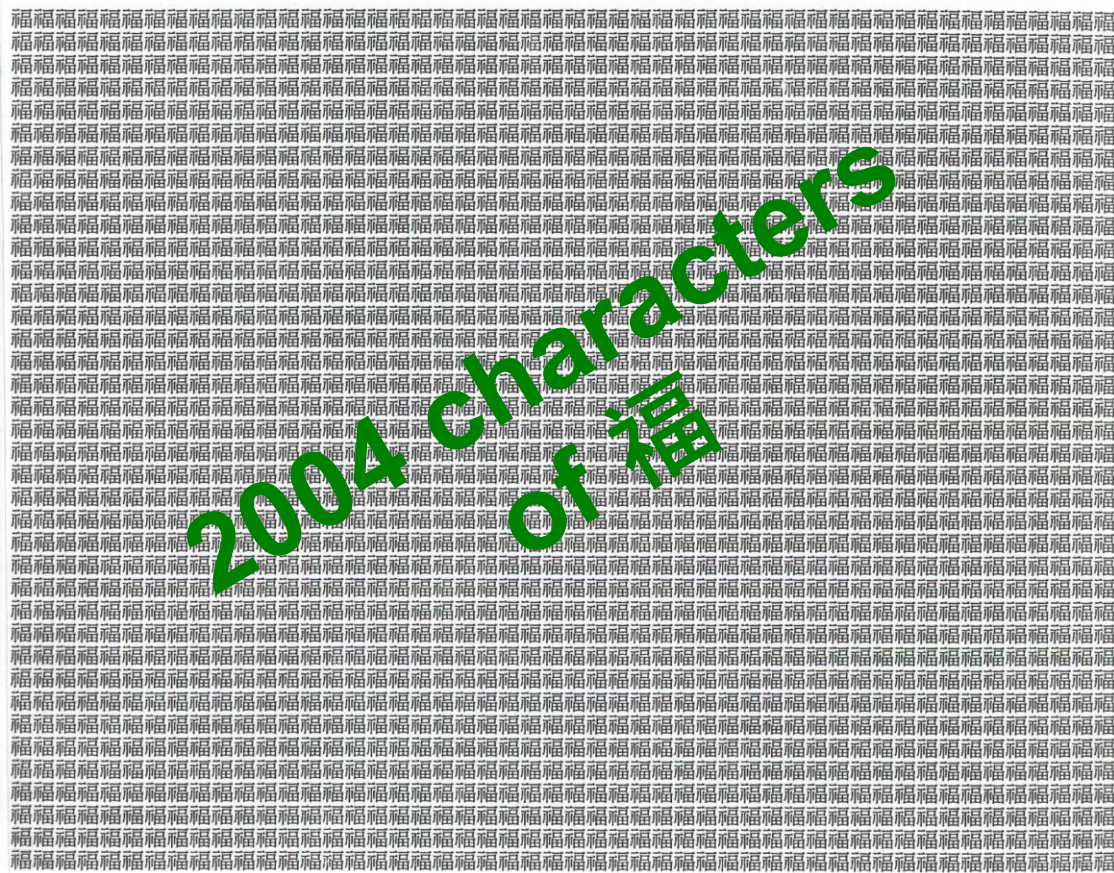


**Jean le Rond d'Alembert
(1717-1783)**

[quoted by Edward Kasner in : The present problems
of geometry, *Bull. Amer. Math. Soc.* XI (1905), p.285]

On Sat, 17 Jan 2004, Wong Ngai Ying wrote:

這裡共有2004個福字代表著今年很幸福，希望您轉寄給你其他的好朋友，祝你好運 !!



Dear NY,

Thank you very much for the good wish for happiness. However, I do **not** think there are 2004 characters there (why?)

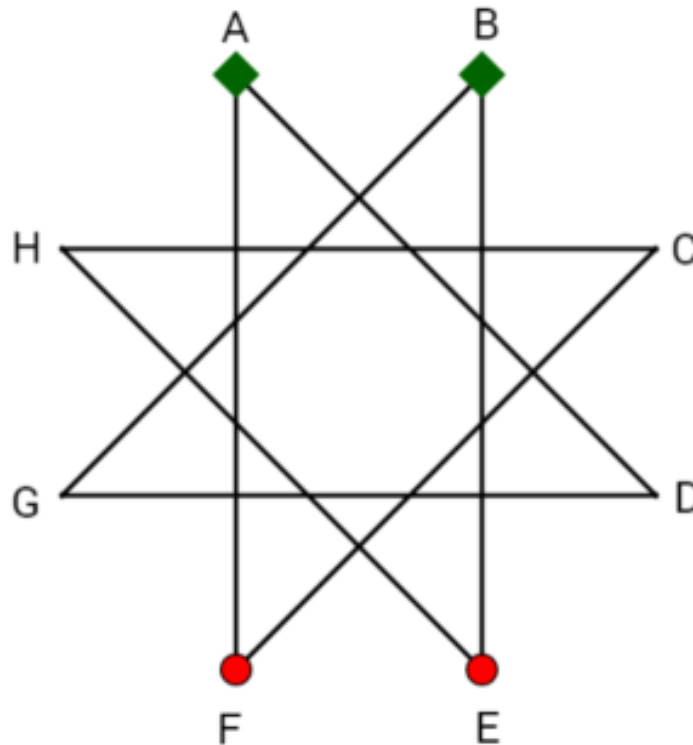
MK

January 17, 2004.

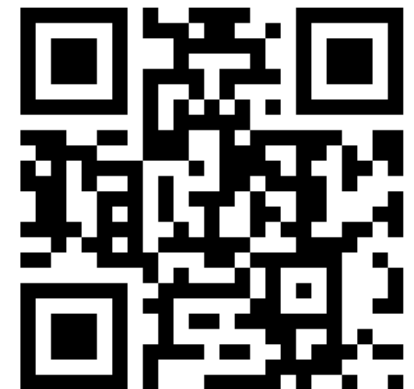
Without counting how do I know there cannot be 2004 characters there?

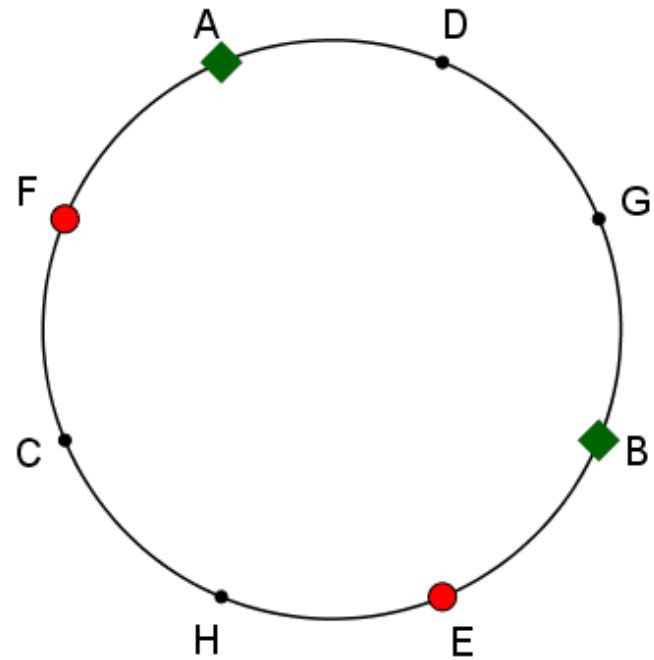
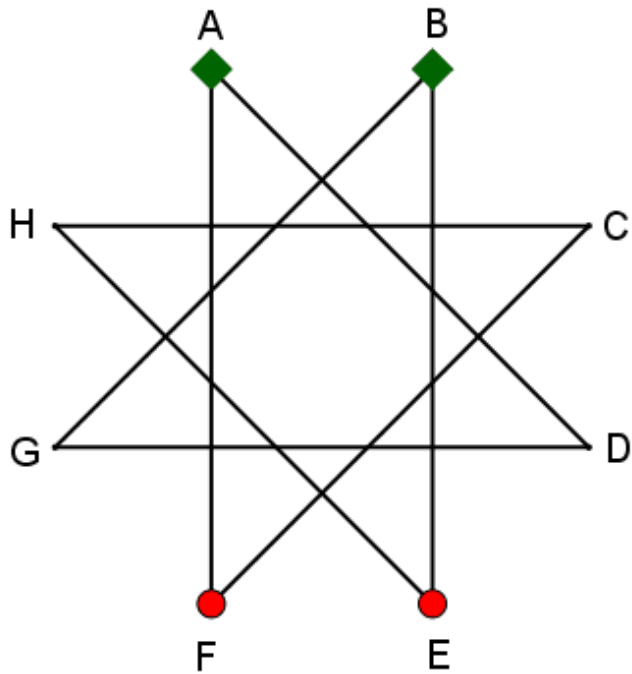
Rule: Each piece can be moved along a straight line to an unoccupied position.

Aim: To interchange the positions of the coloured pieces.



<https://ggbm.at/7182500>





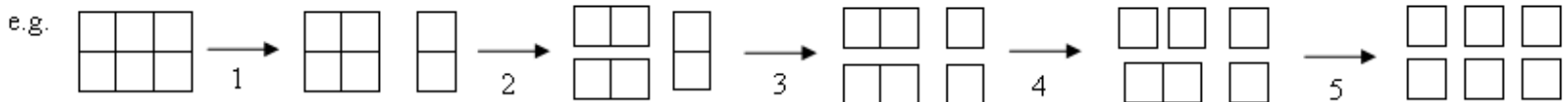
Isomorphic Games



(a) How many football matches need to be played in a **knock-out** competition with **10** teams to produce the champion (where no draw is allowed) ?
What is the answer in the **general case** with **N** teams ?

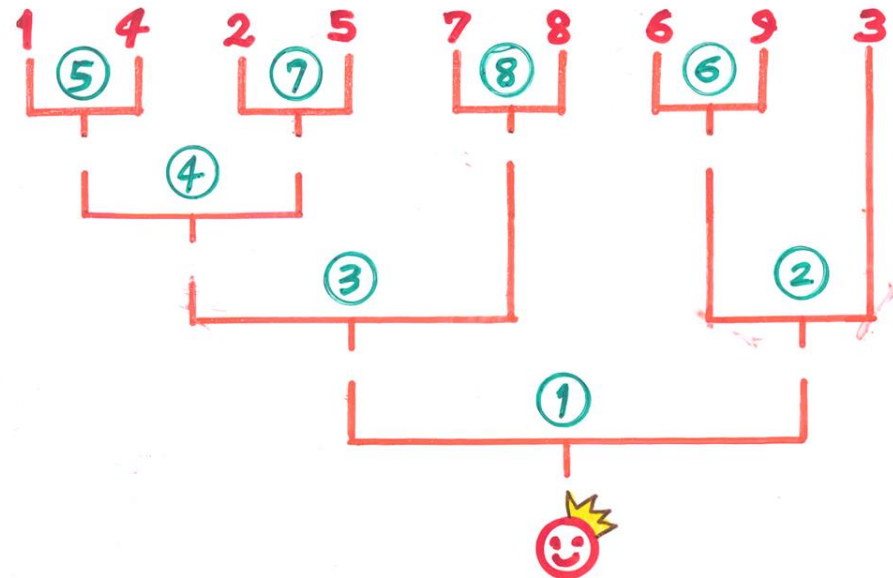
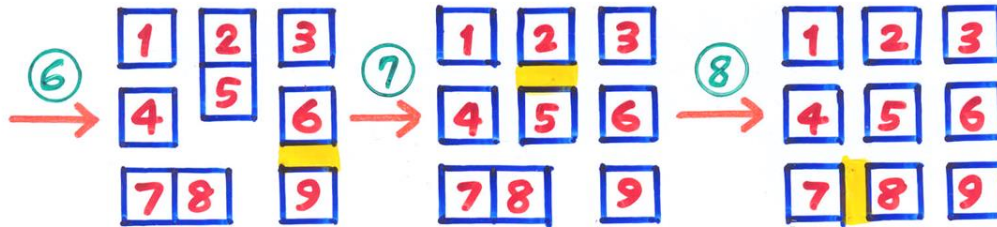
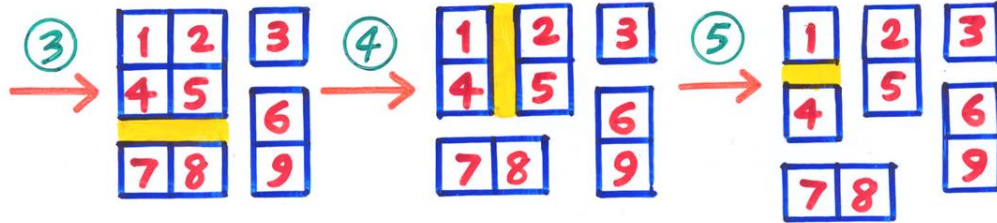
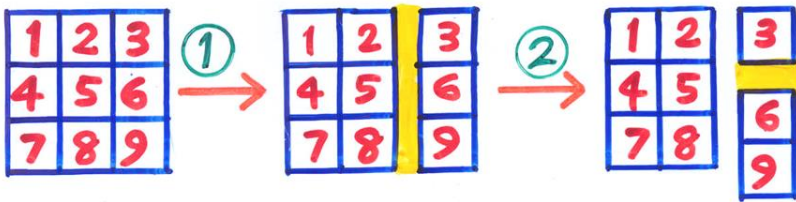


(b) How many times do you need to break up a rectangular pack of chocolate bar with m rows and n columns to get all mn separate pieces?





Are the two problems
in (a) and (b)
essentially the same?
Explain your answer.



Isomorphic Problems

Shuffle a pack of cards, with **half of them face-up and half of them face-down**.

Divide the pack in half with **the same number of cards face-up and face-down in each half**.



4 face-up
2 face-down



4 face-up
2 face-down

A magic card trick invented by Bob Hummer

[See: Chapter 1, Martin Gardner, *Mathematics, Magic and Mystery* (1956)]

In general ...

N face-up

N face-down



a face-up
 $N - a$ face-down

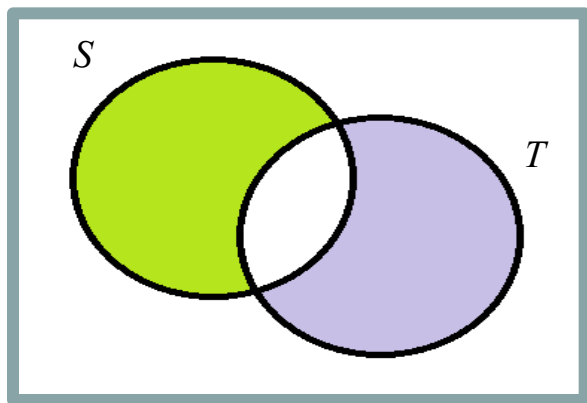
$N - a$ face-up
 a face-down



a face-up
 $N - a$ face down

a face-up
 $N - a$ face-down





If $m(S) = m(T)$,
then $m(S \setminus T) = m(T \setminus S)$.

M = the set of all 52 cards;

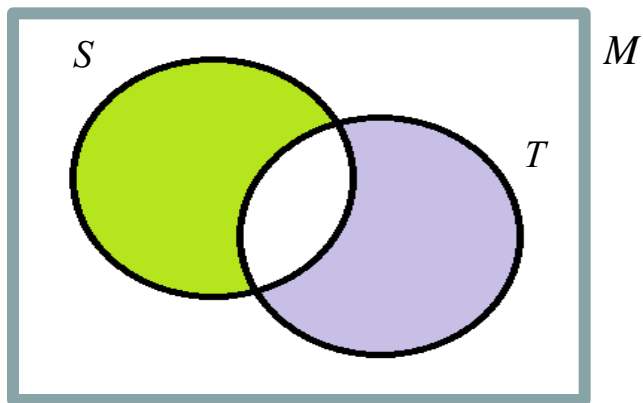
S = the set of 26 face-up cards;

T = the set of 26 cards in one pile (A).

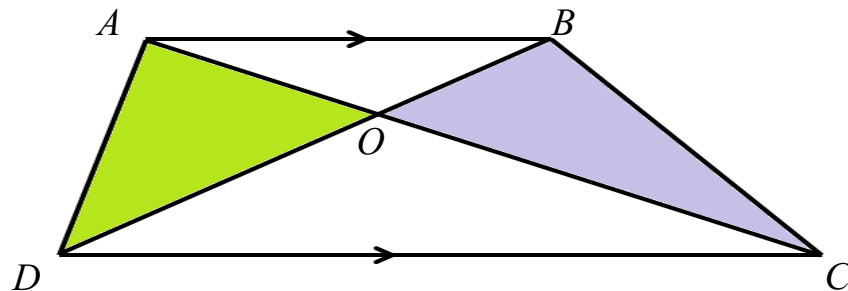
Then $(S \setminus T)$ = the set of face-up cards
in the other pile ($M - A$);

$(T \setminus S)$ = the set of face-down cards
in the pile A .

By reversing one of the two piles,
the number of face-up cards
in each will be the same.



If $m(S) = m(T)$,
then $m(S \setminus T) = m(T \setminus S)$.



$ABCD$ is a trapezium with AB parallel to DC .

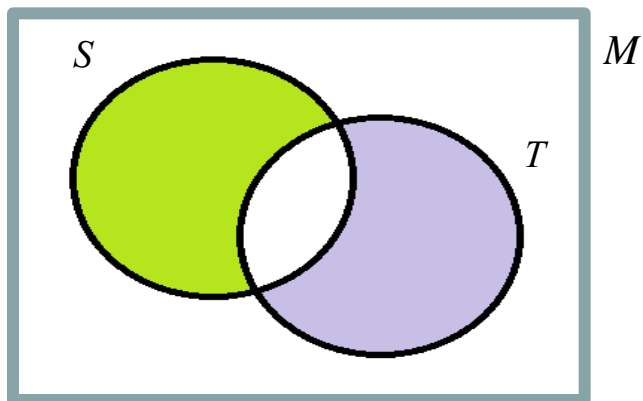
AC and BD intersect at O .

Then $\triangle AOD$ and $\triangle BOC$ have the same area.

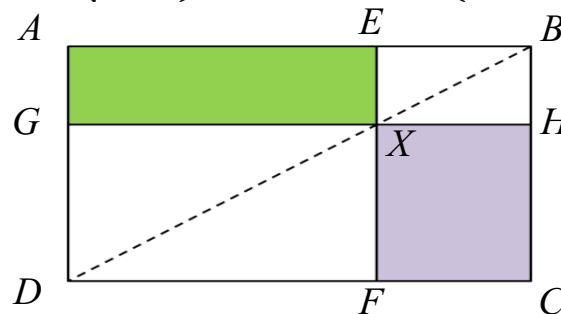
Let $S = \triangle ACD$;

$T = \triangle BCD$.

Since $\triangle ACD$ and $\triangle BCD$ have the same area,
it follows that $\triangle AOD$ and $\triangle BOC$ have the
same area.



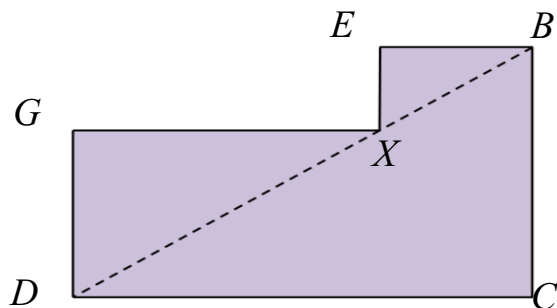
If $m(S) = m(T)$,
then $m(S \setminus T) = m(T \setminus S)$.



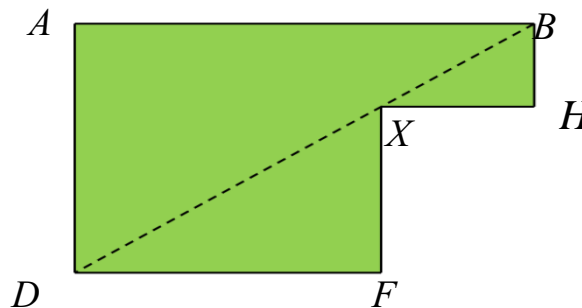
$ABCD$ is a rectangle, and X is a point on the diagonal BD .

EXF is a parallel to AD ; GXH is parallel to AB .

Then the rectangles $AEXG$ and $XHCF$ have equal area.



Let $S = CDGXEB$



Let $T = ABHXFD$

Since $CDGXEB$ and $ABHXFD$ have the same area, it follows that $AEXG$ and $XHCF$ have the same area.

勾(股)中容橫。股(勾)中容直。

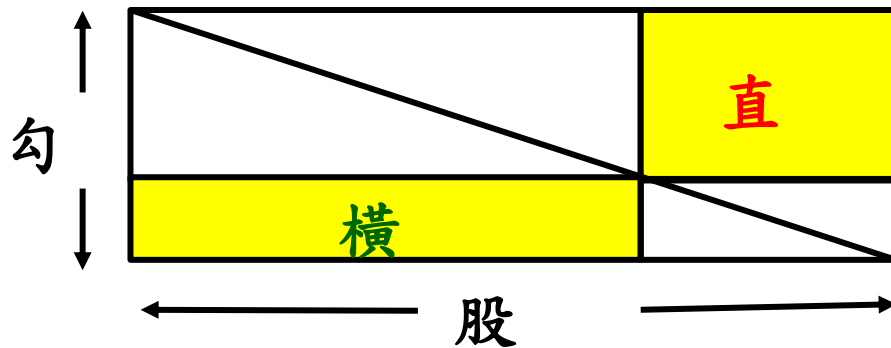
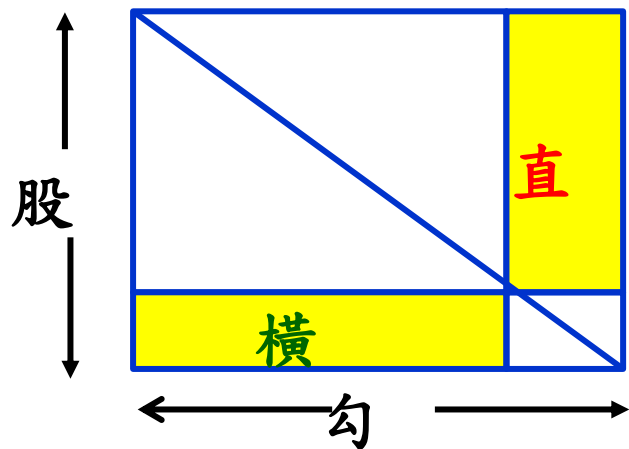
二積皆同。古人以題易名。

若非釋名。則無以知其源。

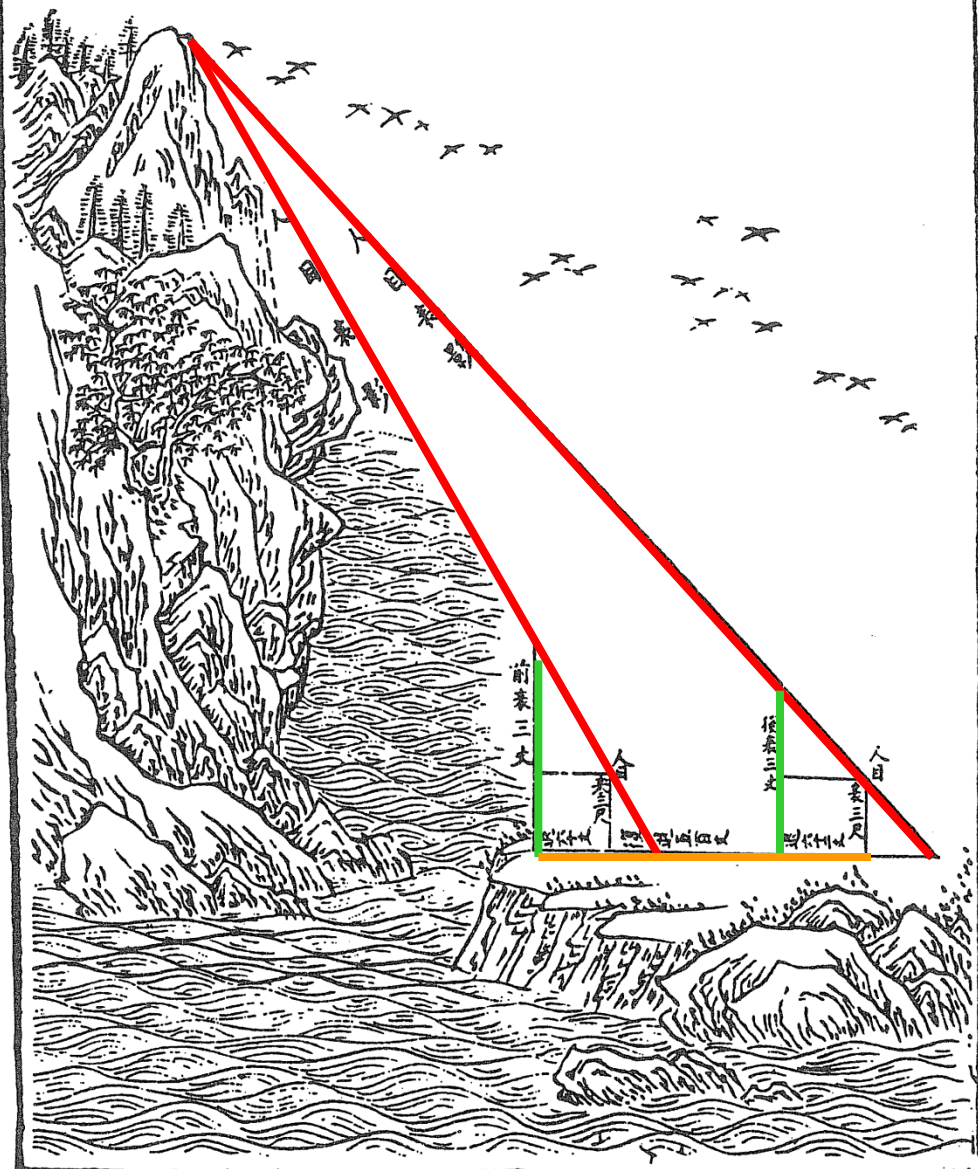
(The horizontal rectangle formed by part of the base and the vertical rectangle formed by part of the perpendicular are equal in area. Men of the past changed the names of their methods from problem to problem ...)

Compare with Proposition 43 of Book I of Euclid's *Elements*.

楊輝，《續古摘奇算法(卷下)》
YANG Hui, *Continuation of Ancient Mathematical Methods for Elucidating the Strange [Properties of Numbers]* (Chapter II) (1275)



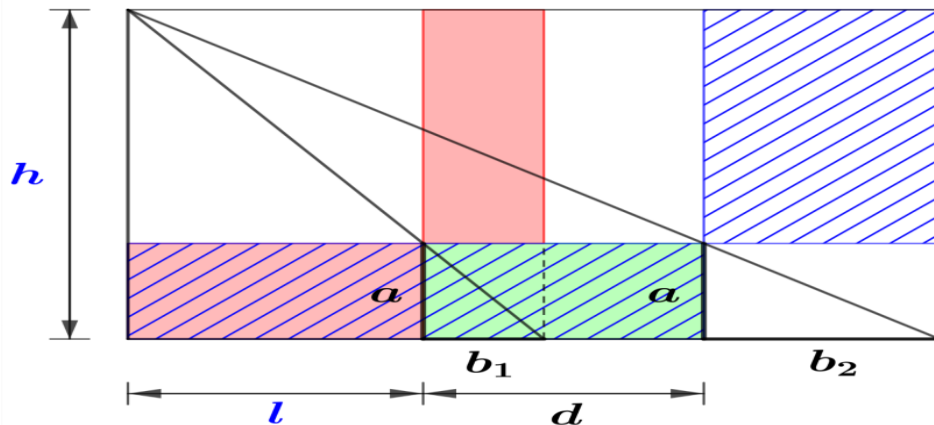
國之島海望窺



Method of Double-Difference

of LIU Hui [劉徽]
in *Haidao Suanjing*
[海島算經 Sea Island
Mathematical Manual]
(3rd century)
as illustrated in
Gujin Tushu Jicheng
[古今圖書集成 Complete
Collection of Pictures
and Writings of
Ancient and
Modern Times]
(1726)

Given a , d , b_1 and b_2 , how can we express h and l in terms of a , d , b_1 and b_2 ?



$$\begin{aligned} \text{Green rectangle} &= \text{Blue hatched rectangle} - \text{Red rectangle} \\ &= \text{Blue hatched rectangle} - \text{Red rectangle} \end{aligned}$$

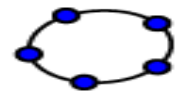
$$\begin{aligned} \therefore ad &= (h - a)b_2 - (h - a)b_1 \\ &= (h - a)(b_2 - b_1) \end{aligned}$$

$$h = \frac{ad}{b_2 - b_1} + a$$

$$\begin{aligned} \text{Red rectangle} &= \text{Red rectangle} \\ \therefore la &= b_1(h - a) = \frac{b_1ad}{b_2 - b_1} \end{aligned}$$

$$l = \frac{b_1d}{b_2 - b_1}$$

<http://ggbtu.be/m2812113>



Explanation by YANG Hui on the
Method of Double-Difference of LIU Hui
(1275)

Aryabhata I 阿耶波多

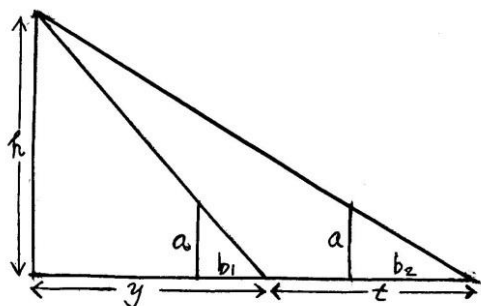
(c.476-550)

Aryabhatiya 《阿耶波多曆數表》



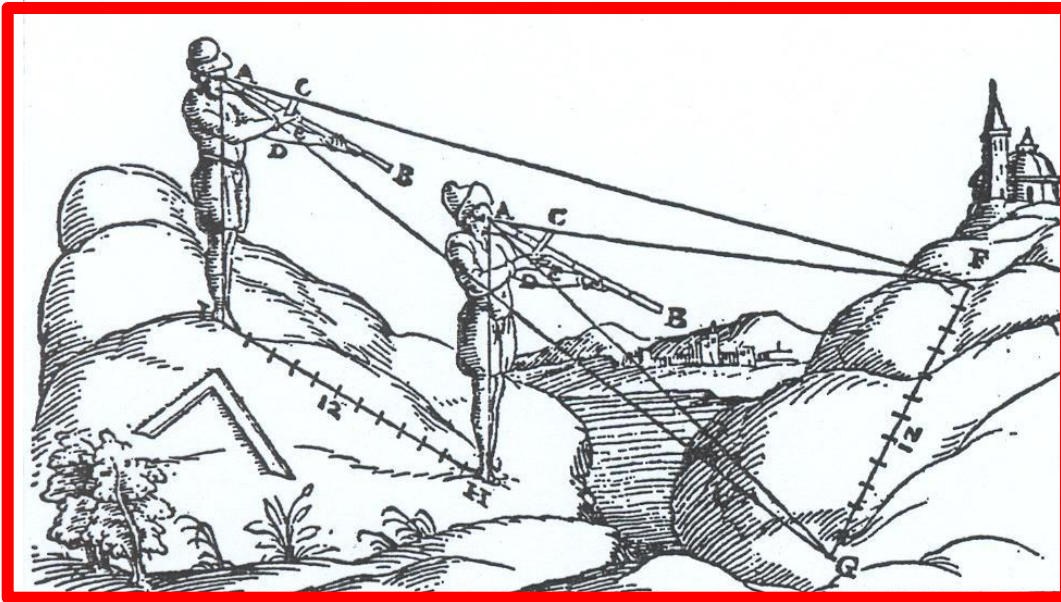
Book II, Stanza 16

The distance between the ends of the two shadows multiplied by the length of the first shadow and divided by the difference in length of the two shadows gives the *kotī*. The *kotī* multiplied by the length of the gnomon and divided by the length of the (first) shadow gives the length of the *bhujā*.



$$y = \frac{tb_1}{b_2 - b_1}$$

$$h = \frac{ya}{b_1} \left(= \frac{ta}{b_2 - b_1} \right)$$



Orence Fine, *De re
& praxi geometrica* (1556)

John Sellers,
Practical Navigation
(1672)



A Jacob's staff, from John Sellers'
Practical Navigation (1672)

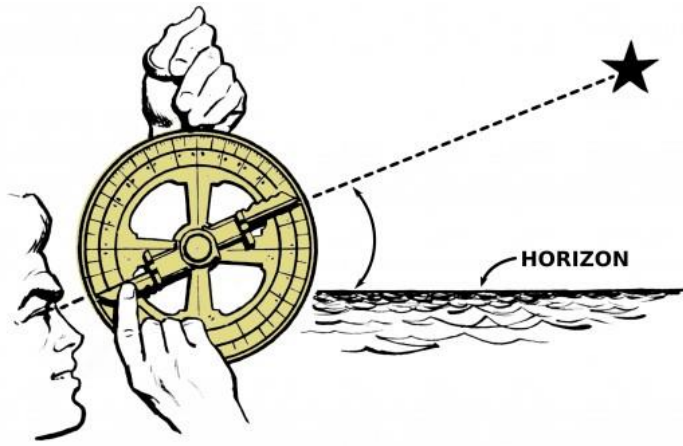
The invention of the cross-staff
(or Jacob's staff) has been credited to
Levi ben Gerson (1288-1344).



**Statue of Al-Biruni in Laleh Park,
Tehran, Iran.**

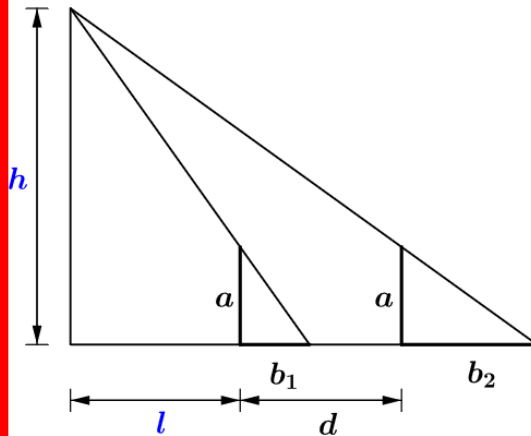
**Abū Rayhān
Muhammad
ibn Ahmad
Al-Bīrūnī
[usually known
As Al-Biruni]
(973-1048)**





Measuring (inaccessible) height with an astrolabe

Given a , d , b_1 and b_2 , how can we express h and l in terms of a , d , b_1 and b_2 ?

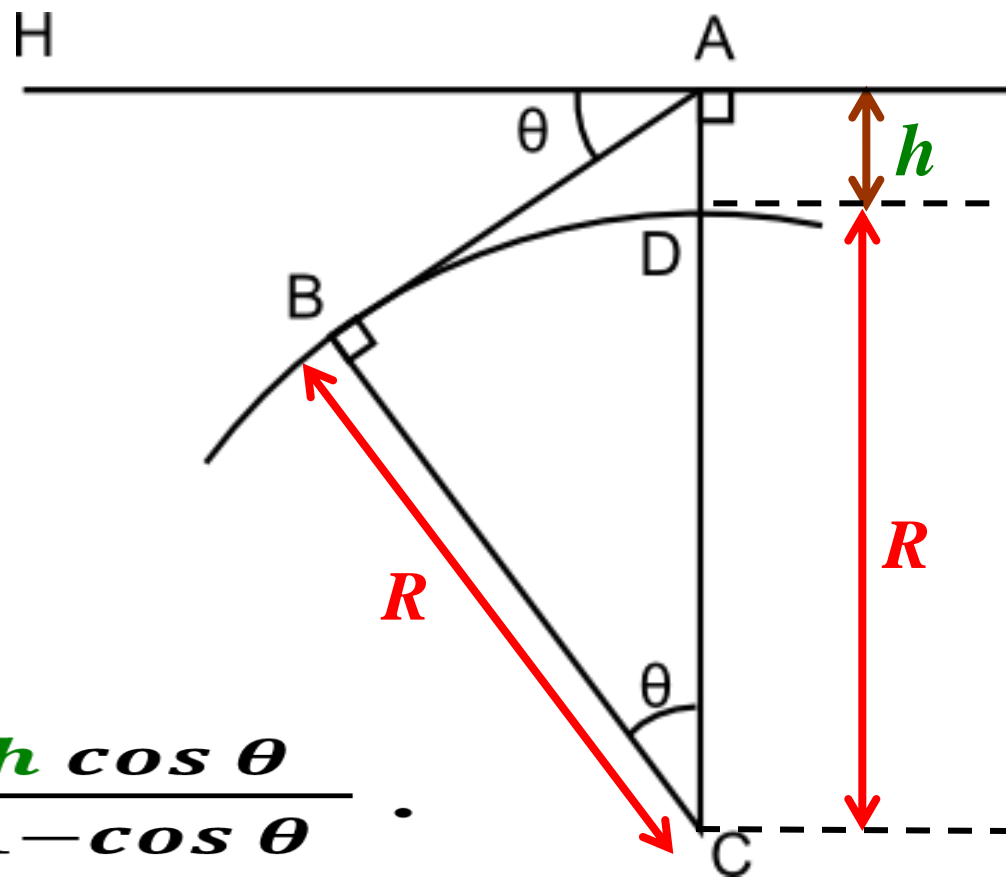




Al-Biruni (973-1048)

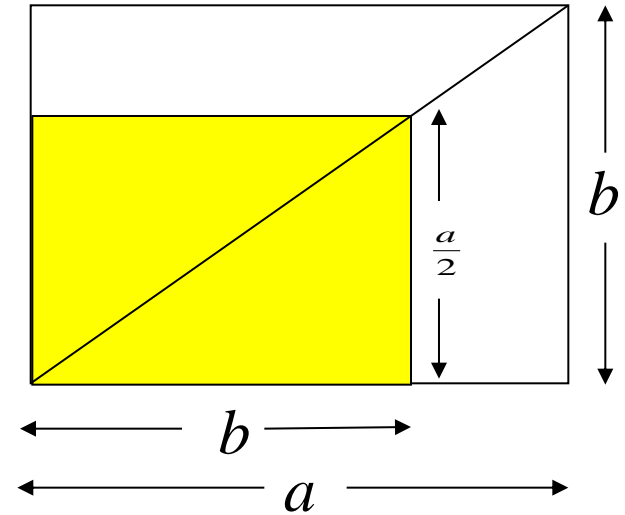
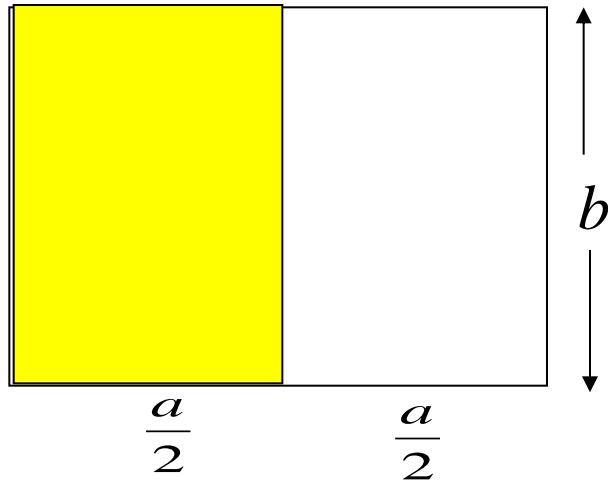
$$\frac{R}{h+R} = \cos \theta ,$$

$$\text{Hence, } R = \frac{h \cos \theta}{1 - \cos \theta} .$$



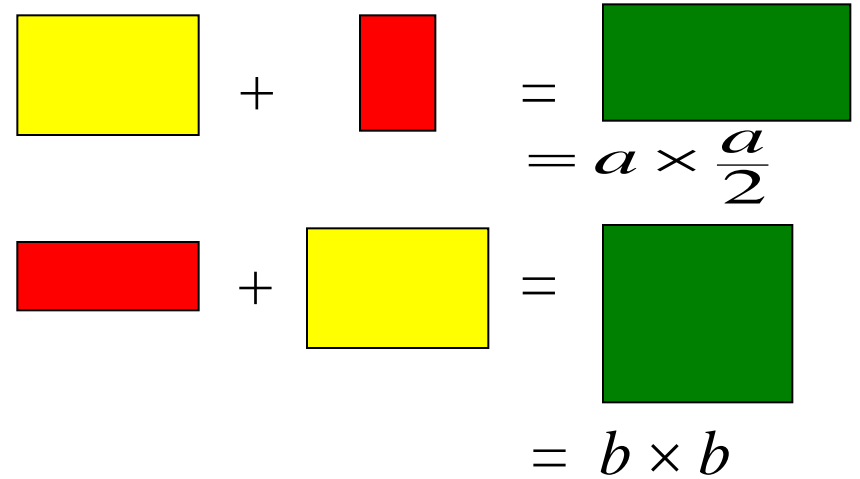
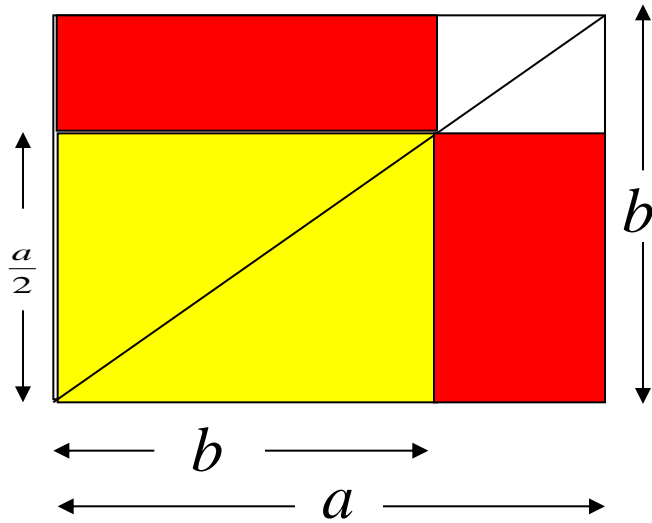
Measurement of the size of the earth by Al-Biruni :
First measure the **height h of a mountain.** From the top of the mountain measure the **angle of depression θ of the horizon.** Then compute the radius of the earth **R** by using trigonometry.

Size of A3 and A4 paper



$$a = ? , b = ? .$$

Factor of enlargement and reduction in a photocopier



$$a \times \frac{a}{2} = b \times b$$

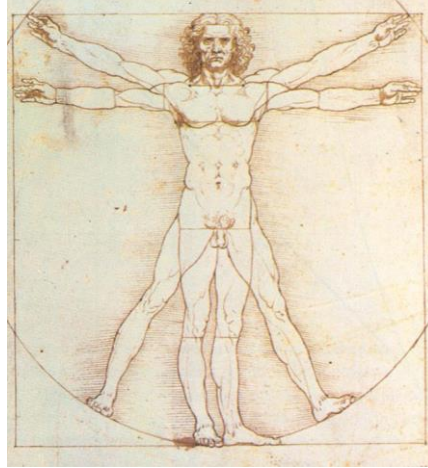
$$a \times a = 2 \times b \times b$$

$$a = \sqrt{2} \times b$$

Magnifying factor = $(\sqrt{2}) \times 100 \% \approx 141 \%$

Shrinking factor = $(1/\sqrt{2}) \times 100 \% \approx 71 \%$

A man weighing **50Kg** can normally lift up **30Kg** .
How much can a man weighing **100Kg** normally lift up?



↑ Weight (W) is proportional to the **cube** of height.
 H ↓ Weight capable of lifting up (F) is proportional to the cross-sectional area of the muscle, hence proportional to the **square** of height (H).

$$\frac{W_1}{W_2} = \frac{H_1^3}{H_2^3}, \quad \frac{F_1}{F_2} = \frac{H_1^2}{H_2^2},$$

Therefore $\frac{W_1^2}{W_2^2} = \frac{F_1^3}{F_2^3}$.

If $W_1 = 2W_2$, then $\frac{F_1^3}{F_2^3} = 4$, $\frac{F_1}{F_2} = \sqrt[3]{4} = 1.5874\dots$

Since F_2 is **30Kg** , F_1 is computed to be **47.62...Kg** .

An ant normally measures **0.005m**. It can carry a burden that is **5 times its own weight**. If a giant ant were as big as a man (say of height **1.75m**), **how much times of its own weight** would it be able to carry?



$$H_2$$

$$H_1 = \frac{1.75}{0.005} H_2 = 350H_2$$

$$W_1 = 350 \times 350 \times 350 \times W_2, \quad F_1 = 350 \times 350 \times F_2,$$

It is known that $\frac{F_2}{W_2} = 5,$

therefore
$$\frac{F_1}{W_1} = \frac{350 \times 350 \times F_2}{350 \times 350 \times 350 \times W_2}$$

$$= \frac{1}{350} \times \frac{F_2}{W_2} = \frac{1}{70},$$

The giant ant can only carry 1/70 of its own weight. It can hardly stand on its own feet!

Allocation problems

occur frequently in everyday life, for example, sharing the expense of a meal, allocating work among a group of people, sharing a prize or goodies, apportioning seats of legislators for different districts,
Is it easy? Is it difficult?

Does knowledge in division alone suffice?

The Lion's Share

The Lion went once a-hunting along with the Fox, the Jackal, and the Wolf. They hunted and they hunted till at last they surprised a Stag, and soon took its life. Then came the question how the spoil should be divided. "Quarter me this Stag," roared the Lion; so the other animals skinned it and cut it into four parts. Then the Lion took his stand in front of the carcass and pronounced judgment: "The first quarter is for me in my capacity as King of Beasts; the second is mine as arbiter; another share comes to me for my part in the chase; and as for the fourth quarter, well, as for that, I should like to see which of you will dare to lay a paw upon it."

"Humph," grumbled the Fox as he walked away with his tail between his legs; but he spoke in a low growl.

"You may share the labours of the great,
but you will not share the spoil."

Æsop's Fables 伊索寓言
(c. 600 B.C.)



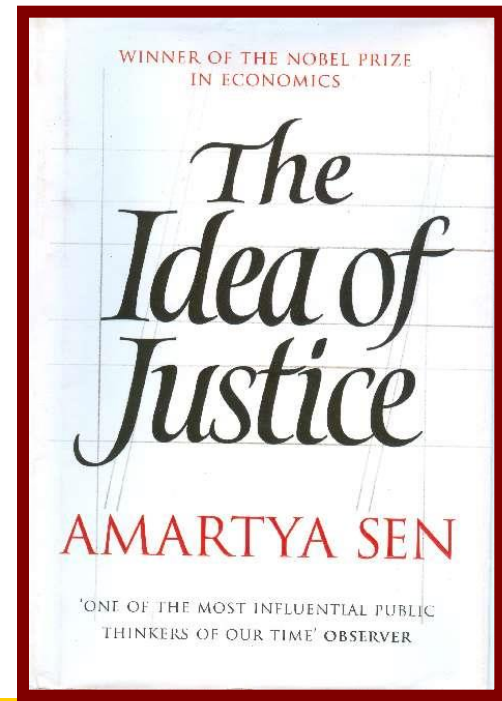
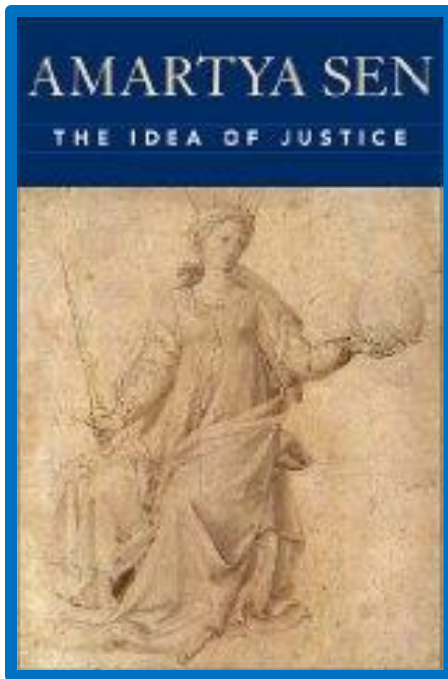
Principles of Allocation

分配原則

❖ 公開 *Open*

❖ 公平 *Fair*

❖ 公正 *Just*



The Idea of Justice

Amartya Sen (2009)



Amartya Kumar Sen (1933-)
Nobel Memorial Prize in Economic Sciences, 1998

A, B, C shared a taxi to go home. A's fare was \$20, B's fare was \$60 and C's fare was \$80. A and B got off before C, and C paid \$80 when he reached home. Putting aside friendly generosity **how should the expense be shared?**

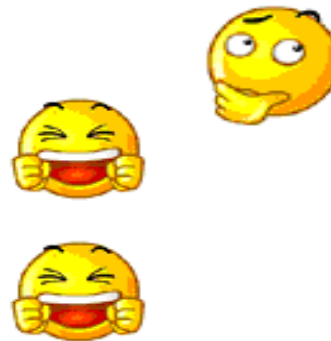
Plan 1 :

Shared the expense **equally.**

A paid \$26.70 ,

B paid \$26.70 ,

C paid \$26.70 .



20



20



40



20



40



20



A, B, C shared a taxi to go home. A's fare was \$20, B's fare was \$60 and C's fare was \$80. A and B got off before C, and C paid \$80 when he reached home. Putting aside friendly generosity **how should the expense be shared?**

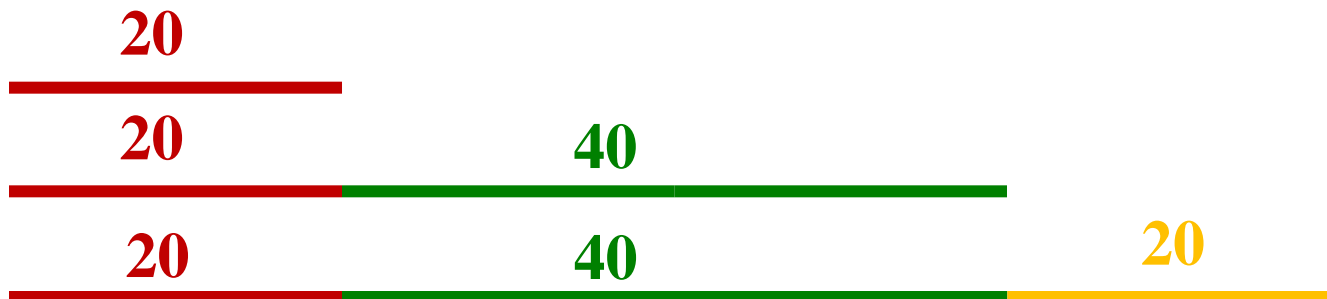
Plan 2 :

By sharing a taxi they **saved** $\$160 - \$80 = \$80$, so this sum should be divided **equally** and be **deducted** from what each should have paid if taking a taxi alone.

A paid - $\$6.60$ (i.e. **gained \$6.60!**),

B paid $\$33.40$,

C paid $\$53.40$.



A, B, C shared a taxi to go home. A's fare was \$20, B's fare was \$60 and C's fare was \$80. A and B got off before C, and C paid \$80 when he reached home. Putting aside friendly generosity **how should the expense be shared?**

Plan 3 :

Shared by proportion **20:60:80.**

A paid $\$80 \times 1/8 = \10 ,

B paid $\$80 \times 3/8 = \30 ,

C paid $\$80 \times 4/8 = \40 .



20



20



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20



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20



A, B, C shared a taxi to go home. A's fare was \$20, B's fare was \$60 and C's fare was \$80. A and B got off before C, and C paid \$80 when he reached home. Putting aside friendly generosity **how should the expense be shared?**

Plan 4 :

A paid $\$20 \times 1/3 = \6.70 ,

B paid $\$20 \times 1/3 + \$40 \times 1/2$
 $= \$26.70$,

C paid $\$20 \times 1/3 + \$40 \times 1/2 + \$20$
 $= \$46.70$.



20

20

20

40

40

20

A, B, C shared a taxi to go home. A's fare was \$20, B's fare was \$60 and C's fare was \$80. A and B got off before C, and C paid \$80 when he reached home. Putting aside friendly generosity **how should the expense be shared?**

Plan 4 :

$$\text{A paid } \$20 \times 1/3 = \$6.70 ,$$

$$\text{B paid } \$20 \times 1/3 + \$40 \times 1/2 \\ = \$26.70 ,$$

$$\text{C paid } \$20 \times 1/3 + \$40 \times 1/2 + \$20 \\ = \$46.70 .$$

How to balance and assess the pros and cons of different plans?

A, B, C shared a taxi to go home. A's fare was \$20, B's fare was \$60 and C's fare was \$80. A and B got off before C, and C paid \$80 when he reached home. Putting aside friendly generosity **how should the expense be shared?**

Plan 4 :

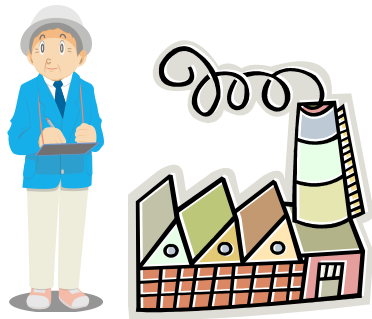
$$\text{A paid } \$20 \times 1/3 = \$6.70 ,$$

$$\text{B paid } \$20 \times 1/3 + \$40 \times 1/2 \\ = \$26.70 ,$$

$$\text{C paid } \$20 \times 1/3 + \$40 \times 1/2 + \$20 \\ = \$46.70 .$$

Is there any other alternative plan?

On what basis is the sharing of expense carried out?



A



B



C



D

A + B

profit of \$100 m.

A + B + C

profit of \$500 m

A + B + D

profit of \$500 m

A + B + C + D

profit of \$2000 m

No **A**, there is no factory.

No **B**, the factory cannot operate.

How should the profit of \$2000m be shared fairly?

A: I own the factory. If there is no factory, there is no job and no profit. Hence I should take \$1000m, B takes \$500m, and C, D each takes \$250m.

B: That is not fair! The factory cannot operate without me. I should get at least as much as A. Split the \$1500m between us (A and me), each getting \$750m. C and D each gets \$250m.

C: That is not fair! I earn for A and B an extra \$400m. I should take \$400m, and so should D. A and B can split the remaining sum, so each gets \$600m.

D: No, we earn for them an extra \$1900m. Each of us (C and me) should take \$950m. Let A and B split the remaining sum, so each gets \$50m.

B: Come on, you do not have your job if there is no factory. You cannot operate without me either. We should at least share that extra \$1900m equally among ourselves, each getting \$475m. Then A and I each gets an extra \$50m, making \$525m.

C: Why don't we simply divide \$2000m equally among ourselves, each getting \$500m?

A: That is not fair, because I own the factory!

A: I own the factory. If there is no factory, there is no job and no profit. Hence I should take \$1000m, B takes \$500m and C, D each takes \$250m.

B: That is not fair! The factory cannot be sold. I should get at least as much as A (A and me), each getting \$750m.

C: That is not fair! I should get \$400m, and B should get \$600m, so each gets \$500m.

D: No! I should get \$500m, and B should get \$500m, so each gets \$500m.

B: You can't sell the factory. — the solution leads to a Nobel Memorial Prize in Economic Sciences of 2012! We should at least share that extra \$500m among ourselves, each getting \$475m.

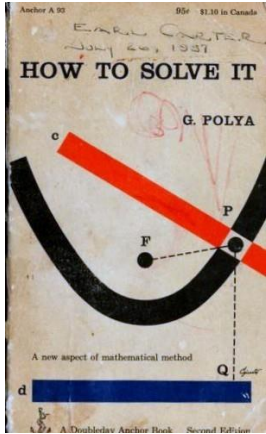
C: Why not? Why don't we simply divide \$2000m equally among ourselves, each getting \$500m?

A: That is not fair, because I own the factory!

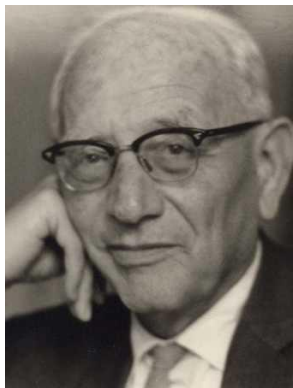
More about this problem in the next lecture

— the solution leads to a Nobel Memorial Prize in Economic Sciences of 2012!

Steps in Problem Solving à la Polya



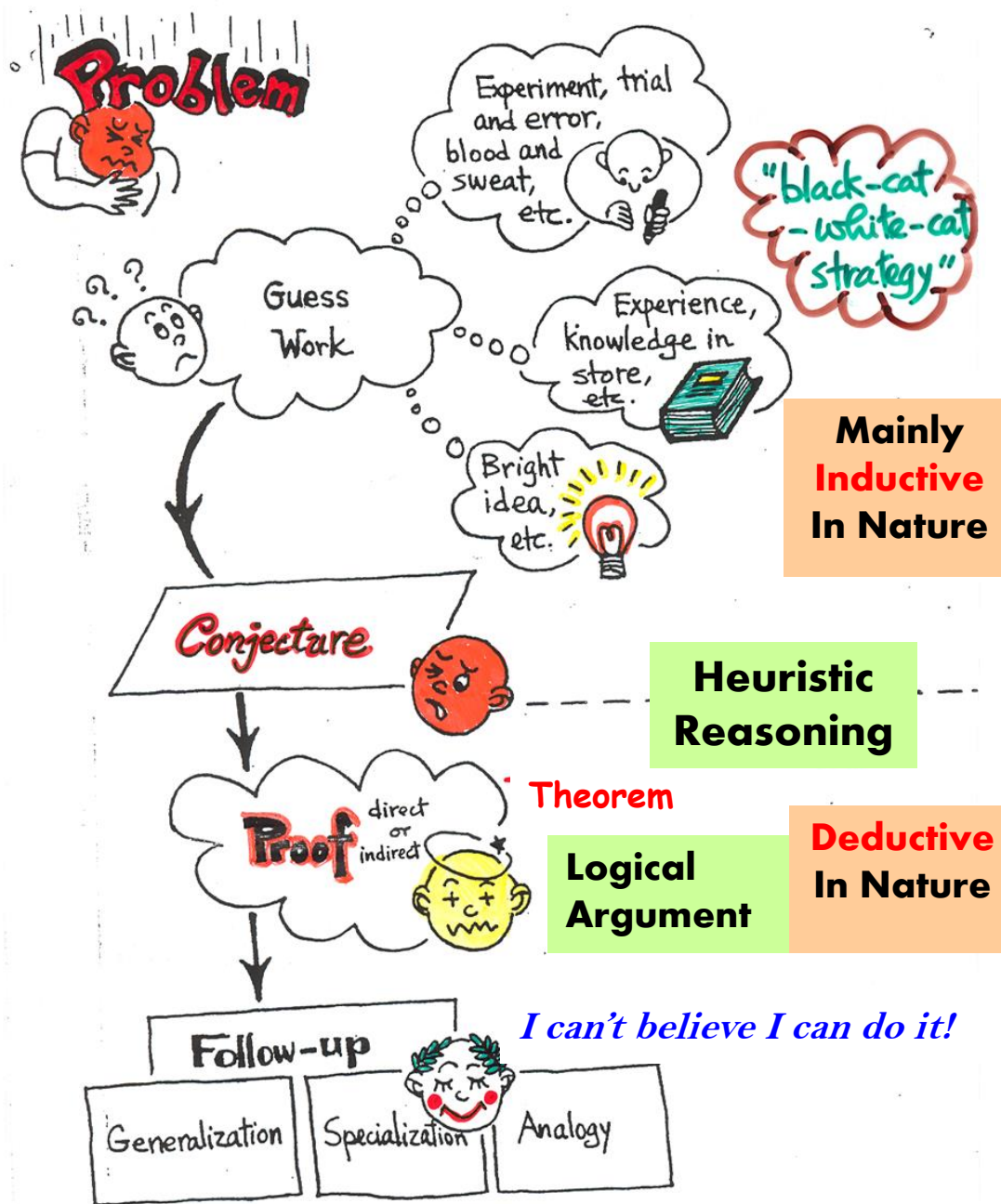
George Pólya
How To Solve It
 1957 Edition
 (first published in 1945)



George Pólya
 (1887-1985)



1974



Mainly Inductive In Nature

Heuristic Reasoning

Theorem Logical Argument

Deductive In Nature

I can't believe I can do it!

Quod Erat emonstrandum!

Question :

Try to partition N consecutive numbers

$1, 2, 3, \dots, N$

into two subsets with the sum of the numbers in one subset equal to that of the numbers in the other subset.

For precisely which N is this possible?

**Can we partition
1, 2 this way?**

**Can we partition
1, 2, 3 this way?**

**Can we partition
1, 2, 3, 4 this way?**

**Can we partition
1, 2, 3, 4, 5 this way?**

Question :

$6 = 1 + 2 + 3$ is the sum of some consecutive positive integers, and so is

$$7 = 3 + 4.$$

Try to find all N that can be written as a sum of (at least two) consecutive positive integers.

	1	2	3	4	5	6	7	8
1		3	6	10	15	21	28	36
2			5	9	14	20	27	35
3				7	12	18	25	33
4					9	15	22	30
5						11	18	26
6							13	21
7								15

Each row is the sum of two, three, four, five, consecutive positive integers starting with the number at the head of the column.

Do you notice what numbers are missing?

Question :

N persons went to meddle with N closed lockers in a row. The first person opens all lockers. The second person closed every second locker starting with the 2nd one. The third person changed the state (opened or closed) of every third locker starting with the 3rd one, and so on until the Nth person changed the state of the Nth locker.



At the end which lockers were left open?