Lecture 2

First let's take a look at the following video presentation by Prof. Eugenia Cheng (this video illustrates how an excellent presentation looks like):

https://www.youtube.com/watch?v=MGGKW0HCb5U

Recap of Lecture 1

In the previous lecture, we talked about “addition” of integers, also “multiplication” of integers.

There we discussed

<u>The Concept of “Closed”-ness</u>

(i-a) the concept of “closed-ness”. For example, “addition” is a closed operation for integers.

Why?

Because if n is an integer (i.e. any positive or negative whole number or zero), m is another one, then their sum $n + m$ is also an integer.

So people use the word “closed” to mean that “adding two integers won’t produce some object not in the system (i.e. a non-integer, such as fraction, or real number like pi)

(i-b) $\times$ of integers is (also) a "closed" operation; (any two integers multiplied together is still an integer)

Remark. this correct a mistake in my lecture!

Question. Can you think of example of similar “systems” in your major subject?

Notation

Mathematicians use the notations $(\mathbb{Z}, +)$ to mean the “system” consisting of (i) integers, (ii) addition.

The notation $\mathbb{Z}$ originates from the German word Zahl (pronounced as “ts-aaa-l” and has the meaning of "integers")

<u>The concept of Neutral Element</u>

- In the system of addition & integers (i.e. our $(\mathbb{Z}, +)$ if you like abstract notations), a neutral element (i.e. element which does nothing under "addition"), is the number 0. In the case of multiplication, the element which does nothing, i.e. 1)

Remark

A neutral element is like a "doing nothing" object, so adding it would produce nothing new. Multiplying with it would also produce nothing new.

Mathematicians would write the above as:
$n + 0 = n$ (n stands for integer here!)
And also $r \times 1 = r$ (r stands for fraction he

The Concept of Inverse Element

- Think about our system of <u>integers together with addition</u>. Whenever you give me an integer, say 3, I want to find an integer (called "inverse of 3" which does the following: $3 + inverse(3) = 0$). Now everybody knows that this "inverse(3)" is nothing but "negative 3", written as -3.
  Mathematicians would write like this: Given any integer n, we can always find an inverse of it denoted by the notation $-n$. When $-n$ is added to n a zero is produced. That is $n + (-n) = 0$.

  Remark.

  <u>For the system of rational numbers together with multiplication, i.e. $(\mathbb{Q}, \times)$ similar thing holds</u>, i.e.
- For any given rational number r, there is always an inverse of it denoted by $inv(r)$. That is, $r \times inv(r) = 1$. In school, we learned that this $inv(r)$ is nothing but $1/r$.

  A Catch

  The above underlined line has one mistake in it. Can you find it out?

  Hint: Can we find "inverse of zero"?

The Concept of Group

Any system which satisfies “closed-ness”, “having neutral element” and “having inverses” is called a group

Remarks

- To be more correct, we require something called associativity, i.e.

$$a * (b * c) = (a * b) * c$$

Here $*$ means any operation in the system.

- There are some other very subtle points, which we will omit.

Importance of the Group Conceopt

Why is group important? Isn't it just a game?

Nope. Coz first of all, it is central to the proof that a quintic equation, i.e. an equation like

$$x^5 + 20x^4 - 3x^3 + 17x^2 - x + 1 = 0$$

may not have simple formulas to write down the x.

Remark

This is surprising because in school math, we learned that any quadratic equation, e.g.

$$ax^2 + bx + c = 0$$

has solutions x which can be written by formulas like $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

In the seventeenth and eighteenth centuries, mathematicians managed to show that for cubic (i.e. power 3) equations, e.g.

$$3x^3 + 17x^2 - x + 1 = 0$$

one has similar formulas.

For quartic (i.e. power 4) equation, e.g.

$$10x^4 - 3x^3 + 17x^2 - x + 1 = 0$$

also.

But for quantic (power 5), no such formula is possible!

This was the great achievement of two young mathematicians, Abel and Galois, one from Norway and the other from France.

https://www.youtube.com/watch?v=g7L_r6zw4-c

A Easy-to-Understand (?) Video about Groups
https://www.mathsisfun.com/sets/groups-introduction.html

Exercises (Try them. And if you can't work it out, just send me an e-mail, I will explain to you).

1. Show that the system consisting of the set $\{0\}$ and addition is a group.
2. Show that the system consisting of the set $\{0\}$ and multiplication is a group.
3. Show that the system consisting of the set $\{1\}$ and addition is not a group.
4. Show that the system consisting of the set $\{1\}$ and multiplication is a group.
5. Show that the system consisting of the set $\{-1, 1\}$ is a group under multiplication, but not addition.

For those who are very enthusiastic and would like to know more, you can browse through

http://www.math.harvard.edu/~jjchen/docs/Group%20Theory%20and%20the%20Rubik's%20Cube.pdf