Lecture 4

Group Theory and Geometry

(Keyword: Structuralism)

1. Recap.

Why groups? In the 1st week, we talked about digital number system.

But then turn to other systems, like base 7 or any natural number.

(Everyday example) Base 12 arithmetic

- The multiplication table is "closed" (unlike our multiplication table in school)

(Why mention this e.g.?)

Because it is a simple case of a <u>mathematical concept</u> known as "group".

(Group's greatest contribution)

Insolvability of Quintic (actually more, e.g. non-trisectability of angle, squaring the circle etc.)

- A quantic equation is something like $x^5 + 10x^4 - 7x^3 + x^2 - 21x + 5 = 0$

(Group satisfies certain conditions)

- (Keywords) closed, associative, inverse, neutral element

<u>Many groups are related to "Geometry"</u>.

So what is Geometry for you?

Suggested Solution: Geometry = "measuring the earth"

An Example of School Geom

Let ABC be a triangle with $AB = AC$, show that $\angle ABC = \angle ACB$.

Q: How would you do it?

One student's answer

- Bisect (using compasses and straightedge, i.e. a ruler without any markings like centimeter, inch on it) the angle $\angle BAC$. Give the point where the angle bisector intersects the line BC the name D.
- Then show that the triangle ΔABD is congruent to the triangle ΔACD by using Side-Angle-Side (Why? Can you rewrite the proof yourself?)

In school geometry, a central concept is "congruence".

Q: What is it? How would you describe it?

One way to say this is to use the concept of "overlapping".

That is, you move one triangle to the other until both of them overlap completely.

(Important new word = "move").

We use concepts related to the intuitive idea of "movements" all the time in math, without noticing it.

Eg. $x \to 2x + 3$

This is a "motion", moving the no. x to the new no. $2x + 3$.

- You can also use words like "associate" or "relate" instead of "move".

Coming back to our triangle example.

Q: What is so special about "congruence"?

During the movement, the triangle is not distorted, i.e. neither enlarged, nor shrinked.

In other words, no <u>length change</u>. No <u>angle change</u>!

So school geometry is actually about "congruence", which is about "moving while keeping lengths and angles unchanged.

The word for such kind of length-keeping-motion

isometry (iso = same, metry < metric = measuring)

i.e. same length

(i.e. the object before "moving" and after "moving" have same lengths. Hence same angle, because angle is related to length. But that's another story.)

This is the SAS, ASA, SSS theorems in geometry.

Summary

Q: What have we done just now?

A: Proved a geometry theorem.

Q: Central concept used ?

A: Congruence.

Q: What is congruence about?

A: About isometries.

So "What is Geometry" about?

Let's look at School Geometry again!

Here is an interesting website about School Geometry.

https://mathigon.org/course/euclidean-geometry/introduction

Go to the page “Geometric Constructions” and you can find interactive explanations about “congruence”.

In the following website (top of page 1), you can read how Euclid defined a “point”, a “line” etc.

https://mathcs.clarku.edu/~djoyce/java/elements/bookI/bookI.html

Q: What is a point? Line? Are they clearly defined?

A: No! One can interpret them with lots of freedom.

The Fifth (Parallel) Axiom

Euclid wrote down 5 axioms (Euclid called them “postulates”) about Geometry (read it from the page “Euclid’s Axioms” in the above website).

The fifth one was most controversial! Many people thought that it was a “theorem” rather than an axiom (an “axiom” is a self-evident statement. You don’t need to prove it! A theorem is something you have to proof.)

The catch

The fact that concepts like “point”, “line” are not clearly defined, mathematicians had freedom to find “interpretations” of such concepts (i.e. points, lines) to check whether the fifth axiom is correct or not.

That was the starting point of Non-Euclidean Geometry.

Non-Euclidean Geometry

Look at these pictures of Escher

https://www.google.com/search?q=escher+poincare+disk&client=firefox-b-d&tbm=isch&source=iu&ictx=1&fir=cYih_SZOEDTnDM%253A%252C960YGEkPj9UobM%252C_&vet=1&usg=AI4_-kSqfYdn_nMJt8Et1_ktChB041JXeA&sa=X&ved=2ahUKEwiclKCDxejkAhVBQd4KHc3WDNYQ9QEwAXoECAcQCQ#imgrc=cYih_SZOEDTnDM:

Here, a “line” is defined as either (i) a straight line passing through the center of the disk, or (ii) a circular arc intersecting the circle at 90 degrees.

In this kind of geometry, give a line l and a point p not in the line, there are many lines passing through p and “parallel” to l.

Another example

https://en.wikipedia.org/wiki/Spherical_geometry

Yet another example (projective plane)

https://en.wikipedia.org/wiki/Projective_plane

A projective plane consists of a set of lines, a set of points, and a relation between points and lines called incidence, having the following properties:[2]

> Given any two distinct points, there is exactly one line incident with both of them.
>
> Given any two distinct lines, there is exactly one point incident with both of them.
>
> There are four points such that no line is incident with more than two of them.

The second condition means that there are no parallel lines.

Compare the above with Eulid's axioms

1. To draw a straight line from any point to any point.

2. To produce a finite straight line continuously in a straight line.

3. To describe a circle with any center and distance.

4. That all right angles are equal to one another.

5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Source:
https://www.pitt.edu/~jdnorton/teaching/HPS_0410/chapters/non_Euclid_postulates/postulates.html