MATH 2055
Suggested Solution to homework 3

Q2 I denotes the n-th term of each sequences by a_n

(a) bounded above by $\underset{n\in\mathbb{N}}{sup}\, a_n = 3$, bounded below by $\underset{n\in\mathbb{N}}{inf}\, a_n = -2$, monotone increasing, $\lim_{n\to\infty} a_n = 3$

(b) bounded above by $\underset{n\in\mathbb{N}}{sup}\, a_n = 2$, bounded below by $\underset{n\in\mathbb{N}}{inf}\, a_n = 1$, monotone decreasing, $\lim_{n\to\infty} a_n = 1$

(c) $a_n = \frac{n}{1-\frac{1}{n+1}} = n+1$
no upper bound, bounded below by $\underset{n\in\mathbb{N}}{inf}\, a_n = 2$, monotone increasing, divergent

(d) bounded above by $\underset{n\in\mathbb{N}}{sup}\, a_n = 1$, bounded below by $\underset{n\in\mathbb{N}}{inf}\, a_n = -1$, $\lim_{n\to\infty} a_n = 0$

(e) bounded above by $\underset{n\in\mathbb{N}}{sup}\, a_n = 2$, bounded below by $\underset{n\in\mathbb{N}}{inf}\, a_n = 0$, $\lim_{n\to\infty} a_n = 0$

Q3 (a) by definition, for all $\epsilon > 0$, there exists m, such that $\sup_{n\in\mathbb{N}} a_n b_n - \epsilon < a_m b_m$

also, $0 \leq a_m \leq \sup_{n\in\mathbb{N}} a_n$ and $0 \leq b_m \leq \sup_{n\in\mathbb{N}} b_n$

as ϵ is arbitrary positive number, we have $\sup_{n\in\mathbb{N}} a_n b_n \leq (\sup_{n\in\mathbb{N}} a_n)(\sup_{n\in\mathbb{N}} b_n)$

the equality may not hold.

for example, we can pick

$a_1 = 1000$, $a_n = 1$ for all $n > 1$

$b_1 = 1$, $b_n = 2$ for all $n > 1$

then $\sup_{n\in\mathbb{N}} a_n b_n = 1000$ while $(\sup_{n\in\mathbb{N}} a_n)(\sup_{n\in\mathbb{N}} b_n) = (1000)(2) = 2000$

(b) by definition, for all $\epsilon > 0$, there exists m, such that $\sup_{n\in\mathbb{N}} |a_n + b_n| - \epsilon < |a_m + b_m|$

by triangle inequality, $|a_m + b_m| \leq |a_m| + |b_m|$

also, $|a_m| \leq \sup_{n\in\mathbb{N}} |a_n|$ and $|b_m| \leq \sup_{n\in\mathbb{N}} |b_n|$

as ϵ is arbitrary positive number, we have $\sup_{n\in\mathbb{N}} |a_n + b_n| \leq \sup_{n\in\mathbb{N}} |a_n| + \sup_{n\in\mathbb{N}} |b_n|$

(c) by definition, for all $\epsilon > 0$, there exists m, such that $\sup_{n\in\mathbb{N}} a_n - \epsilon < a_m \leq \sup_{n\in\mathbb{N}} a_n$

which imply $a_m \leq \sup\limits_{n \in \mathbb{N}} a_n < a_m + \epsilon$

as $|a_m| \leq \sup\limits_{n \in \mathbb{N}} |a_n|$ and

$|a_m + \epsilon| \leq |a_m| + \epsilon$

we have $|\sup\limits_{n \in \mathbb{N}} a_n| \leq max\{|a_m|, |a_m + \epsilon|\} \leq |a_m| + \epsilon \leq \sup\limits_{n \in \mathbb{N}} |a_n| + \epsilon$

so $|\sup\limits_{n \in \mathbb{N}} a_n| \leq \sup\limits_{n \in \mathbb{N}} |a_n|$

Q6 (a) prove it by MI.

when $n = 1$, $2 \leq x_1 = 2.5 \leq 3$

assume $2 \leq x_k \leq 3$ for some natural number k

then $4 \leq x_k^2 \leq 9$

and hence $2 \leq a_{k+1} = \frac{1}{5}(x_k^2 + 6) \leq 3$

(b) $\frac{1}{5}(x_n - 2)(x_n - 3) = (\frac{1}{5})(x_n^2 - 5x_n + 6) = x_{n+1} - x_n$

(c) as $2 \leq x_n \leq 3$

$x_n - 2 \geq 0$ and $x_n - 3 \leq 0$

imply $x_{n+1} - x_n \leq 0$

so $\{x_n\}$ is monotone decreasing. As it is bounded below, it is convergent.

let $x = \lim\limits_{n \to \infty} x_n$

then $\lim\limits_{n \to \infty} (x_{n+1} - x_n) = \lim\limits_{n \to \infty} \frac{1}{5}(x_n - 2)(x_n - 3)$

which imply $x = 2$ or $x = 3$
but x_n is decreasing and $x_1 = 2.5$, hence $x = 2$

Q8 there exists N such that for all $n > N$, $a_n < l + 1$

take $M = max\{a_1, a_2, \cdots, a_N, l + 1\}$

then $a_n \leq M$ for all n and hence $\{a_n\}$ is bounded above.

for all $\epsilon > 0$, there exist N such that $a_N > sup(a_n) - \epsilon$

as a_n is increasing, for all $n > N$, $sup(a_n) - \epsilon < a_n \leq sup(a_n) < sup(a_n) + \epsilon$

so we have $\lim_{n\to\infty} a_n = sup(a_n)$

as limit is unique, so we have $l = sup(a_n)$

if a_n is monotone decreasing and convergent, then $l = inf(a_n)$

Q12 (a) by definition, for all $\epsilon > 0$, there exist $m \geq n+1$ such that $a_{n+1} - \epsilon < x_m$

as $m \geq n+1$, we have $x_m \leq a_n$ and hence $a_{n+1} - \epsilon < a_n$

as ϵ is arbitrary positive number, so we have $a_{n+1} \leq a_n$

so a_n is monotone decreasing.
as x_n is bounded sequence, let $|x_n| \leq M$ for all M.

then $-M \leq x_m \leq a_{n+1}$

so a_n is bounded below and hence convergent.

(b) by definition, for all $\epsilon > 0$, there exist $m \geq n+1$ such that $b_{n+1} + \epsilon > x_m$

as $m \geq n+1$, we have $x_m \geq b_n$ and hence $b_{n+1} + \epsilon > b_n$

as ϵ is arbitrary positive number, so we have $b_{n+1} \geq b_n$

so b_n is monotone increasing.
then $M \geq x_m \geq b_{n+1}$

so a_n is bounded above and hence convergent.

(i) $lim\, sup(-1)^n = 1$, $lim\, inf(-1)^n = -1$

(ii) $lim\, sup(\frac{1}{n}) = lim\, inf(\frac{1}{n}) = 0$

(iii) $lim\, sup(-1)^n(1-\frac{1}{n}) = 1$, $lim\, inf(-1)^n(1-\frac{1}{n}) = -1$