MATH 2055
Suggested Solution to homework 2
(Prepared by Ng Wing-Kit)

Q4 Suppose x_n converges to x,
$\forall \epsilon > 0$, $\exists N$, such that $\forall n > N$, $|x_n - x| < \epsilon$

$$\begin{aligned} ||x_n| - |x|| &\leq |x_n - x| \\ &< \epsilon \end{aligned}$$

$\therefore \lim_{n\to\infty} |x_n| = |x|$

Converse is not true. Pick $x_{2m} = 1$ and $x_{2m+1} = -1$ for each natural number m, then (x_n) is divergent while $(|x_n|)$ converges to 1 □

Q5 As $\lim_{n\to\infty} a_n = 0$, by the $\epsilon - N$ definition of limit of sequence,

$\forall \epsilon > 0$, $\exists N$, such that $\forall n > N$, $|a_n - 0| < \epsilon$.

The above inequality implies, in particular that

$$a_n < \epsilon.$$

(we have used one side of the two-sided inequalities $-\epsilon < a_n < \epsilon$).

Next, we estimate the 'distance' of b_n from zero, i.e.

$$\begin{aligned} |b_n - 0| &= b_n \quad (\because 0 \leq a_n \leq b_n) \\ &\leq a_n \\ &< \epsilon \end{aligned}$$

$\therefore \lim_{n\to\infty} b_n = 0$ □

Q7 (a) the condition need to be satisfied for all $\epsilon > 0$

(b) This statement is confusing. If there is a " , " between "for some natural number N" and " where $n > N$ " there is no problem. "for some natural number N where $n > N$ " means that we have n first and then pick a particular N depending on n

(c) It is correct, or more precisely, within ϵ neighbourhood of x

(d) N is not defined and the sentence means that the following condition only true for n in a subset of $\{n|n > N\}$

(e) “ for some ϵ ” $\longrightarrow$ “ for all ϵ ”
n is not defined when the statement define N
if the sequence is not convergent, n may not exist and for all N , $N < n$ automatically true. $\square$

Q8 (a) “ridiculous convergence ” is stronger than the usual convergence

$\exists N$ such that $\forall \epsilon > 0$, $|x_n - x| < \epsilon$ whenever $n > N$

$\Longrightarrow \forall n > N$, $x_n = x$

$\Longrightarrow x_n$ converge to x

(b) $\forall N$, $N + 1 > N$,

$|\frac{1}{N+1} - 0| = \frac{1}{N+1} > \frac{1}{N}$

$\therefore (\frac{1}{n})$ is not ridiculous converge to 0 $\square$

Q9 Replace ϵ in the definition by $C\epsilon$.

Q12 for all natural number m,

$\frac{m-1}{m} \leq \frac{m-cos(m)}{m} \leq \frac{m+1}{m}$

let $a_m = \frac{m-1}{m}$ and $b_m = \frac{m+1}{m}$

$\forall \epsilon > 0$, $\forall n > \frac{1}{\epsilon}$

$|a_n - 1| = \frac{1}{n} < \epsilon$

$|b_n - 1| = \frac{1}{n} < \epsilon$

$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1$

as $a_n \leq \frac{n-cos(n)}{n} \leq b_n$

$\therefore \lim_{n\to\infty} \frac{n-cos(n)}{n} = 1$ $\square$

Q14 (a) As $r > 1$,for all natural number m , if $r^{\frac{1}{m}} \leq 1$, then $r \leq 1^m$ lead to contradiction.
$\therefore r^{\frac{1}{m}} > 1$

let $r^{\frac{1}{m}} = 1 + c_m$ where $c_m > 0$
$(1 + c_m)^m = r$

$\therefore mc_m \leq r - 1$

$\forall \epsilon > 0$,

$\forall n > \frac{r-1}{\epsilon}$,

$|r^{\frac{1}{n}} - 1| = c_m$

$\leq \frac{r-1}{n}$

$< \epsilon$

$\therefore \lim_{n\to\infty} r^{\frac{1}{n}} = 1.$ $\square$

(b) As $0 < r < 1$,for all natural number m , if $r^{\frac{1}{m}} \geq 1$, then $r \geq 1^m$ leads to contradiction.
$\therefore r^{\frac{1}{m}} < 1$

let $r^{\frac{1}{m}} = \frac{1}{1+s_m}$ where $s_m > 0$
$\frac{1}{(1+s_m)^m} = r$

$\therefore ms_m \leq \frac{1}{r} - 1$

$\forall \epsilon > 0,$

$\forall n > \frac{\frac{1}{r}-1}{\epsilon},$

$|r^{\frac{1}{n}} - 1| = \frac{s_n}{1+s_n}$

$< s_n$

$\leq \frac{\frac{1}{r}-1}{n}$

$< \epsilon$

$\therefore \lim_{n\to\infty} r^{\frac{1}{n}} = 1.$ $\square$

(c) for all natural number $m > 1$, let $m^{\frac{1}{m}} = 1 + b_m$ where $b_m > 0$
$(1 + b_m)^m = m$

$C_2^m (b_m)^2 \leq m$

$(b_m)^2 \leq \frac{2}{m-1}$

$\forall \epsilon > 0,$

$\forall n > \frac{2}{\epsilon^2} + 1,$

$|n^{\frac{1}{n}} - 1| = b_n$

$\leq \sqrt{\frac{2}{n-1}}$

$< \epsilon$

$\therefore \lim_{n\to\infty} n^{\frac{1}{n}} = 1.$ $\square$